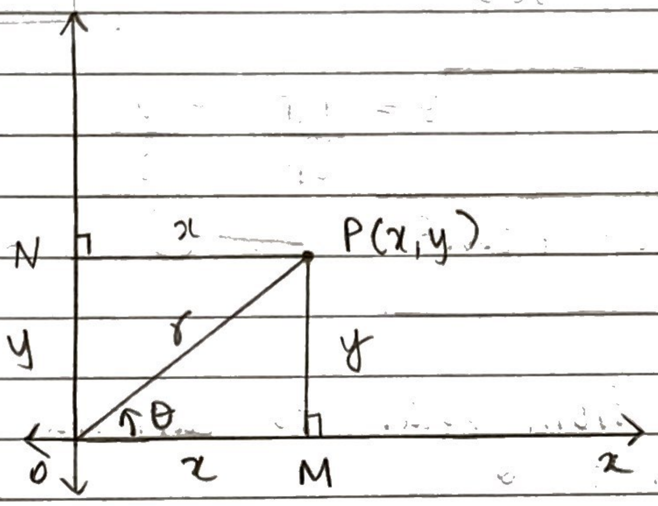
DIFFERENTIAL CALCULUS

Polar Curves

Polar Coordinates



- Let $P(x,y)$ be any point on the plane.
- Draw $PM \perp x$ axis and $PN \perp y$ axis.
- Join OP. Let $|\overrightarrow{OP}| = r$ and $\angle MOP = \theta$
- The real number r is called the radial distance or the radius vector.

- The real number θ is called the radial angle or the vectorial angle.

- The numbers r & θ are called the polar coordinates of P.

- The point O is called the pole. The horizontal line OX is called the initial line or polar axis.

Any curve specified by the equation

$$r = f(\theta)$$

is called a polar curve.

From the figure, $\cos\theta = \frac{OM}{OP} = \frac{x}{r}$

$$\boxed{x = r\cos\theta} \longrightarrow (1)$$

$\sin\theta = \frac{MP}{OP} = \frac{y}{r}$

$$\boxed{y = r\sin\theta} \longrightarrow (2)$$

Transforming from polar to cartesian is simple; ~~replace~~ eliminate θ.

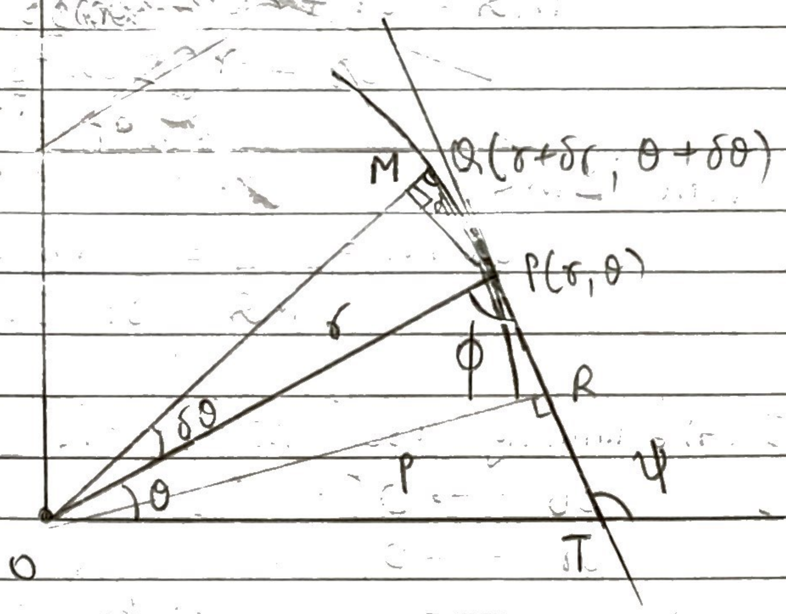
From the above equations,

$$\boxed{r = \sqrt{x^2+y^2}} \longrightarrow (3)$$

$$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)} \longrightarrow (4)$$

(1), (2), (3) & (4) are called transformations.

Angle between radius vector & tangent



Consider a polar curve whose equation is $r = f(\theta)$. Let $P(r,\theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points on the curve.

Let the tangent at P meet the initial line at T, and make an angle ψ with it.

Draw $PM \perp OQ$. Let $\angle OPT = \phi$ and $\angle MQP = \alpha$

From the right angled triangle ~~ΔQ~~ ΔOPM:

$$\sin(\delta\theta) = \frac{PM}{OP} = \frac{PM}{r}$$

$$\boxed{PM = r\sin\delta\theta}$$

&

$$\cos(\delta\theta) = \frac{OM}{OP} = \frac{OM}{r}$$

$$\boxed{OM = r\cos\delta\theta}$$

From the above values,

$$MQ = OQ - OM$$
$$= r + \delta r - r\cos\delta\theta$$
$$MQ = \delta r + r(1-\cos\delta\theta)$$
$$= \delta r + r\left(2\sin^2\frac{\delta\theta}{2}\right)$$

From ΔMQP,

$$\tan\alpha = \frac{PM}{MQ} = \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\frac{\delta\theta}{2}}$$

In the limiting case as $Q \longrightarrow P$,

$\delta\theta \longrightarrow 0$

$\delta r \longrightarrow 0$

$\alpha \longrightarrow \phi$ (chord PQ becomes tangent at P)

$$\therefore \lim_{\alpha\to\phi} \tan\alpha = \lim_{\delta\theta\to 0} \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\left(\frac{\delta\theta}{2}\right)}$$

$$\tan\phi = \lim_{\delta\theta\to 0} \frac{r\,\cancel{\frac{\sin\delta\theta}{\delta\theta}}\,1}{\frac{\delta r}{\delta\theta} + r\left(\sin\frac{\delta\theta}{2}\right)\cancel{\frac{\left(\sin\frac{\delta\theta}{2}\right)}{\frac{\delta\theta}{2}}}\,1}$$

$$\tan\phi = \lim_{\delta\theta\to 0} \frac{r(1)}{\frac{dr}{d\theta} + r(0)(1)}$$

$$\tan\phi = \frac{r}{dr/d\theta} = r\frac{d\theta}{dr}$$

$$\boxed{\tan\phi = r\frac{d\theta}{dr}} \longrightarrow \text{EQUATION 1.1}$$

From the figure, $\psi = \theta + \phi$

$\therefore$ slope of the tangent at P is

$$\tan\psi = \tan(\theta+\phi)$$

From ΔOPR (right angle)

$$\sin\phi = \frac{OR}{OP} = \frac{p}{r}$$

$$\boxed{p = r\sin\phi}$$

The dist. p is called the pedal of the curve, and the relation $p = r\sin\phi$ is called the pedal equation of the curve.

The above equation can also be written as

$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}(1+\cot^2\phi)$$

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2}\left(1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right)}$$

Pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

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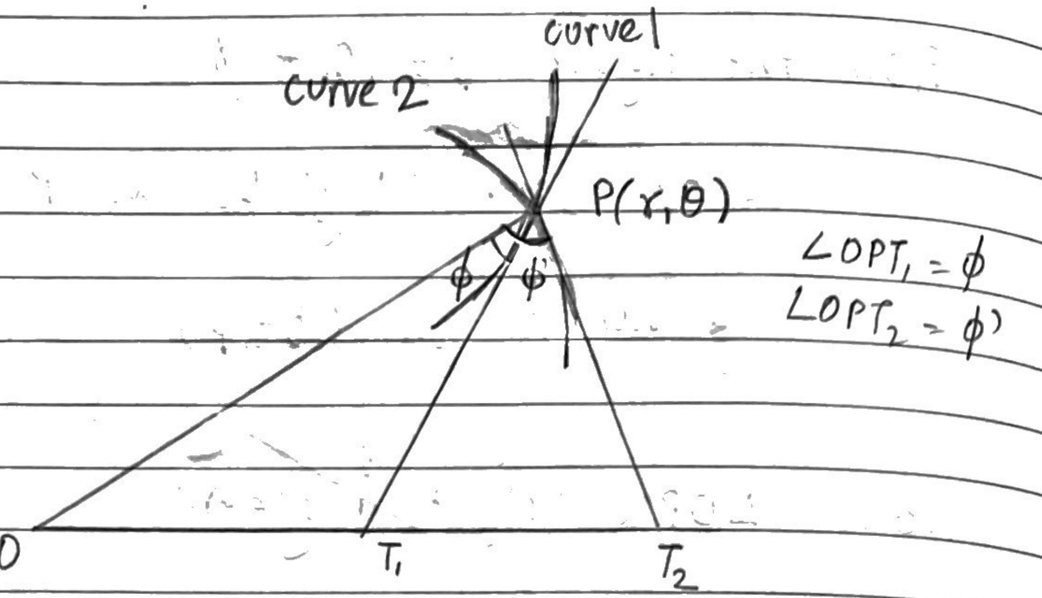
16.08.19

Angle between polar curves.



Let $P(r, \theta)$ be the point of intersection of two polar curves.

Let PT_1 & PT_2 be the tangents to the curves at P.

Let $\angle OPT_1 = \phi$ & $\angle OPT_2 = \phi'$

The angle between the two curves is the same as the angle between the tangents T_1 & T_2.

The required angle from the figure is

$$\angle T_1PT_2 = \phi' - \phi$$

From EQUATION 1.1

$$\angle OPT_1 = \phi = \tan^{-1}\left(r\frac{d\theta}{dr}\right) \text{ of curve 1}$$

$$\angle OPT_2 = \phi' = \tan^{-1}\left(r\frac{d\theta}{dr}\right) \text{ of curve 2}$$

Problems

1. Prove that, in the parabola $\frac{2a}{r} = 1-\cos\theta$,
$\phi = \pi - \frac{\theta}{2}$.

$\cos 2x = 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$

Ans: $\tan\phi = \frac{r}{dr/d\theta}$; $r = \frac{2a}{1-\cos\theta}$

$$\therefore \frac{dr}{d\theta} = 2a \frac{d}{d\theta}\left(\frac{1}{1-\cos\theta}\right) = \cancel{2}a \frac{d}{d\theta}\left(\frac{1}{\cancel{2}\sin^2\theta/2}\right)$$

$$\frac{dr}{d\theta} = a\frac{d}{d\theta} \operatorname{cosec}^2\frac{\theta}{2} = -\frac{1}{\cancel{2}}\cdot a\cdot \cancel{2}\operatorname{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2}$$

$$\frac{dr}{d\theta} = -\operatorname{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2}\cdot a$$

$$\tan\phi = \frac{-r}{\operatorname{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2}\, a} = \frac{-\cancel{2a}\, \cancel{\operatorname{cosec}^2\theta/2}}{\cancel{2a}\cdot\cancel{\operatorname{cosec}^2\frac{\theta}{2}}\cot\frac{\theta}{2}}$$

$$\cancel{\tan\phi = -\tan\frac{\theta}{2} = \tan(}$$

$$\tan\phi = -\tan\frac{\theta}{2} = \tan\left(\pi - \frac{\theta}{2}\right)$$

$$\cancel{\tan\phi + \tan\frac{\theta}{2} = 0}$$

$$\boxed{\phi = \pi - \frac{\theta}{2}}$$

Exam sol.

Given $\frac{2a}{r} = 1-\cos\theta$

$$r = \frac{2a}{1-\cos\theta} = \frac{\cancel{2}a}{\cancel{2}\sin^2\frac{\theta}{2}}$$

$$r = a\,\mathrm{cosec}^2\frac{\theta}{2}$$

Differentiating r wrt θ,

$$\frac{dr}{d\theta} = -\frac{2a}{2}\mathrm{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2} = -a\,\mathrm{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2}$$

$$\tan\phi = \frac{r}{-a\,\mathrm{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2}} = \frac{\cancel{a\,\mathrm{cosec}^2\theta/2}}{-\cancel{a}\,\cancel{\mathrm{cosec}^2\frac{\theta}{2}}\cot\frac{\theta}{2}}$$

$$\tan\phi = -\tan\frac{\theta}{2} = \tan\left(\pi-\frac{\theta}{2}\right)$$

$$\phi = \pi-\frac{\theta}{2}$$

Hence proved.

2. Find the angle between radius vector and the tangent to the curve

$$r^m = a^m(\cos m\theta + \sin m\theta)$$

Differentiating both sides,

$$\cancel{m} r^{m-1} \frac{dr}{d\theta} = a^m(-\cancel{m}\sin m\theta + \cancel{m}\cos m\theta)$$

$$\frac{r^m}{r}\frac{dr}{d\theta} = a^m(\cos m\theta - \sin m\theta)$$

$$\frac{\cancel{a^m}(\cos m\theta + \sin m\theta)}{r}\frac{dr}{d\theta} = \cancel{a^m}(\cos m\theta - \sin m\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta} \longrightarrow 1$$

$\tan\phi$ = angle b/w radius & tangent.

$= \frac{r}{dr/d\theta}$ = reciprocal of eq. 1.

$$\frac{r\,d\theta}{dr} = \frac{\cos m\theta + \sin m\theta}{\cos m\theta - \sin m\theta} = \tan\phi$$

Dividing num & den. by $\cos m\theta$

$$\frac{r\,d\theta}{dr} = \frac{1+\tan m\theta}{1-\tan m\theta} = \tan\left(\frac{\pi}{4} + m\theta\right)$$

$$\tan\phi = \tan\left(\frac{\pi}{4} + m\theta\right)$$

$$\boxed{\phi = \frac{\pi}{4} + m\theta}$$

The angle b/w ~~curve~~ tangent & radius is $\frac{\pi}{4} + m\theta$

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3. Find the angle b/w radius vector & tangent to curve $r\sec^2\frac{\theta}{2} = 4$ at point $\theta = \pi/2$. Also find slope of tangent at this point.

Let ϕ = angle b/w radius vector & tangent,
$\theta = \pi/2$
ψ = angle b/w tangent & axis.

We know $\tan\phi = r\frac{d\theta}{dr}$

Given curve : $r\sec^2\frac{\theta}{2} = 4$

$$r = 4\cos^2\frac{\theta}{2}$$

Differentiating both sides

$$\frac{dr}{d\theta} = (4)(\cancel{2}\cos\frac{\theta}{2})(-\sin\frac{\theta}{2})(\frac{1}{\cancel{2}})$$

$$= -4\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$\frac{dr}{d\theta} = -2\sin\theta$$

$$\tan\phi = \frac{r}{dr/d\theta} = \frac{-r}{2\sin\theta} = \frac{-\cancel{4}\cos^2\theta/2}{\cancel{4}\cancel{\cos\theta/2}\sin\theta/2}$$

$$\tan\phi = -\cot\theta/2 = -\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\tan\phi = \tan\left(\pi - \frac{\pi}{2} + \frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$=$

$\phi = \frac{\pi}{2} + \frac{\theta}{2}$ = angle b/w radius vector & tangent.

$\cancel{\psi = \phi + \theta = \frac{\pi}{2} + \frac{\theta}{2} + \theta}$

But $\theta = \pi/2$

$\therefore \phi = \frac{\pi}{2} + \frac{\pi}{4} = \boxed{\frac{3\pi}{4}}$

Angle b/w rad. & tan. $= 3\pi/4$

$\psi = \phi + \theta = \frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$

Slope $= \tan\psi = \tan\left(\pi + \frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$

$\boxed{\text{Slope} = 1}$

4. For the curve $r^3 = a^3 (\cos 3\theta)$, show that the normal at any point (r, θ) to the curve makes an angle 4θ with the initial line.

$\tan\phi = r\frac{d\theta}{dr}$, Taking nat log on both sides.

$\cancel{3r^2 \frac{dr}{d\theta}} =$

$\cancel{3\log r}$

$$3 \ln r = 3 \ln a + \ln(\cos 3\theta)$$

Dif. both sides.

$$\frac{3}{r}\frac{dr}{d\theta} = \frac{1}{\cos 3\theta} \cdot -3\sin 3\theta.$$

$$\cancel{3}\cot\phi = -\cancel{3}\tan 3\theta$$

$$\cancel{\tan}\ \cot\phi = \tan(\pi - 3\theta)$$

$$= \cot\left(\frac{\pi}{2} - \pi + 3\theta\right)$$

$$\cot\phi = \cot\left(3\theta - \frac{\pi}{2}\right)$$

$$\phi = 3\theta - \pi/2$$

Angle between tangent & normal = $\pi/2$

$\therefore$ angle between radius & normal

$$= \phi + \pi/2 = 3\theta - \pi/2 + \pi/2$$

$$= 3\theta$$

$$\cot\phi = -\tan 3\theta$$

$$= \cancel{\cot}\left(\frac{\pi}{2} + 3\theta\right)$$

$$\phi = \frac{\pi}{2} + 3\theta.$$

$$\psi = \phi + \theta = 4\theta + \pi/2$$

Slope of normal = $-\cot\psi = -\cot(\pi/2 + 4\theta)$

$$= \frac{a.\ -1}{\tan(\pi/2 + 4\theta)} = \frac{+1}{+\cot 4\theta} = \tan 4\theta$$

Slope of normal = tan 4θ.

$\psi' = 4\theta$ = angle b/w normal & initial line (axis)

5. Find the angle of intersection of the curves.
$\underbrace{r = 3\cos\theta}_{\text{curve 1}}$ and $\underbrace{r = 1+\cos\theta}_{\text{curve 2.}}$

Let ϕ = angle b/w tangent & radius of $r = 3\cos\theta$
and ϕ' = angle b/w tangent & radius of $r = 1+\cos\theta$

* we need to find point of intersection to find exact angle.

Eliminating r from curve 1 & curve 2:

$3\cos\theta = 1 + \cos\theta$
$2\cos\theta = 1 \Rightarrow \cos\theta = 1/2$
$\therefore \boxed{\theta = \pi/3} \Rightarrow r = 1+\cos\theta = \boxed{3/2}$

$\tan\phi = \dfrac{r}{dr/d\theta}$ $\qquad$ $\tan\phi' = \dfrac{r}{dr/d\theta}$

Differentiating curve 1 wrt θ
$\dfrac{dr}{d\theta} = -3\sin\theta$

Differentiating curve 2 wrt θ
$\dfrac{dr}{d\theta} = -\sin\theta$

$\tan\phi = \dfrac{\cancel{3}/2}{-\cancel{3}\sin\theta} = \dfrac{-1}{2\sin\pi/3} = -1 \times 2$

$$\tan\phi = \frac{-1}{\sqrt{3}} \quad (\tan \text{ is -ve ; } 2^{nd} \text{ quad})$$

$$\phi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\tan\phi' = \frac{r}{dr/d\theta} = \frac{3/2}{-\sin\theta} = \frac{-3 \times \cancel{2}}{\cancel{2} \times \sqrt{3}} = -\sqrt{3}$$

$$\tan\phi' = -\sqrt{3} \quad (\tan \text{ is -ve, } 2^{nd} \text{ quad})$$

$$\phi' = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Angle between the two curves

$$\phi - \phi' = \frac{5\pi}{6} - \frac{2\pi}{3} = \frac{5\pi}{6} - \frac{4\pi}{6} = \frac{\pi}{6}$$

$$\boxed{\phi - \phi' = \pi/6}$$

Hence, the angle between the two curves is $\pi/6$

Exam sol:

find general solution & then specific.

6. Find the angle between the curves
$r^n = a^n \sec(n\theta \pm \alpha) \longrightarrow C1$ and
$r^n = b^n \sec(n\theta + \beta) \longrightarrow C2$
where $\alpha > \beta$, n, a, b, α, β are constants

~~to C1~~ Taking ln of C1 & C2.

$n \ln r = n \ln a + \ln(\sec(n\theta + \alpha)) \rightarrow C'1$
$n \ln r = n \ln b + \ln(\sec(n\theta + \beta)) \rightarrow C'2$

Diff. C'1

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{\cancel{\sec(n\theta+\alpha)}(\tan(n\theta+\alpha))(n)}{\cancel{\sec(n\theta+\alpha)}}$$

$$\frac{\cancel{n}}{\tan\phi} = \cancel{n}\tan(n\theta+\alpha) = \cot\phi$$

$$\tan\left(\frac{\pi}{2} - \phi\right) = \tan(n\theta + \alpha)$$

$$\frac{\pi}{2} - \phi = n\theta + \alpha$$

$$\boxed{\phi = \frac{\pi}{2} - n\theta - \alpha} \rightarrow (1)$$

Diff. C'2.

$$\frac{\cancel{n}}{r}\frac{dr}{d\theta} = \frac{\cancel{\sec(n\theta+\beta)}\tan(n\theta+\beta)\,\cancel{n}}{\cancel{\sec(n\theta+\beta)}}$$

$$\cot\phi' = \tan(n\theta+\beta)$$
$$\tan\left(\frac{\pi}{2} - \phi'\right) = \tan(n\theta+\beta)$$

$$\frac{\pi}{2} - \phi' = n\theta + \beta$$

$$\boxed{\phi' = \frac{\pi}{2} - n\theta - \beta} \rightarrow (2)$$

Angle b/w curves

$= ② - ① = \phi' - \phi$

$= \frac{\pi}{2} - \cancel{n\theta} - \beta + \cancel{n\theta} + \alpha - \frac{\pi}{2}$

$\boxed{\phi' - \phi = \alpha - \beta}$

7. Show that the curves
$r^2 \cos(2\theta - \alpha) = a^2 \sin 2\alpha \rightarrow C1$ and
$r^2 = 2a^2 \sin(2\theta + \alpha) \rightarrow C2$ cut at
right angles at the point of intersection

Eliminating r,
Divide C1 by C2

$$\frac{\cancel{r^2}\cos(2\theta - \alpha)}{\cancel{r^2}} = \frac{\cancel{a^2}\sin 2\alpha}{2\cancel{a^2}\sin(2\theta + \alpha)}$$

$2\cos(2\theta - \alpha)\sin(2\theta + \alpha) = \sin 2\alpha.$

$2\sin(2\theta + \alpha)\cos(2\theta - \alpha) = \sin 2\alpha$

~~2 sin~~

$\sin(2\theta + \cancel{\alpha} + 2\theta - \cancel{\alpha}) + \sin(\cancel{2\theta} + \alpha - \cancel{2\theta} + \alpha) = \sin 2\alpha$

$\sin(4\theta) + \cancel{\sin 2\alpha} = \cancel{\sin 2\alpha}$

$\sin 4\theta = 0$

$4\theta = n\pi$

$\theta = \frac{n\pi}{4}$

~~$\theta = 0$ or $\theta = \frac{\pi}{4}$ or $\theta =$~~

$$\theta = \frac{n\pi}{4} \quad (n = 0, 1, 2 \cdots, 8)$$

consider c1; ~~Differentiating it~~

$$r^2 \cos(2\theta - \alpha) = a^2 \sin 2\alpha$$

taking ln.

$$2 \ln r + \ln(\cos(2\theta - \alpha)) = \ln(a^2 \sin 2\alpha)$$

Differentiating

$$\frac{\cancel{2}}{r}\frac{dr}{d\theta} + \frac{-\sin(2\theta - \alpha)(\cancel{2})}{\cos(2\theta - \alpha)} = 0.$$

$$\cot \phi = \tan(2\theta - \alpha)$$

$$\tan\left(\frac{\pi}{2} - \phi\right) = \tan(2\theta - \alpha)$$

$$\frac{\pi}{2} - \phi = 2\theta - \alpha$$

$$\boxed{\phi = \frac{\pi}{2} - 2\theta + \alpha}$$

consider c2:

$$r^2 = 2a^2 \sin(2\theta + \alpha)$$

taking ln on both sides

$$2 \ln r = 2 \ln a^2 + \ln(\sin(2\theta + \alpha))$$

Diff.

$$\frac{\cancel{2}}{r}\frac{dr}{d\theta} = 0 + \frac{\cancel{2}\cos(2\theta + \alpha)}{\sin(2\theta + \alpha)}$$

$$\cot \phi' = \cot(2\theta + \alpha)$$

$$\boxed{\phi' = 2\theta + \alpha}$$

$$\phi' - \phi = 2\theta + \cancel{\alpha} - \frac{\pi}{2} + 2\theta - \cancel{\alpha}$$

$$\phi' - \phi = 4\theta - \frac{\pi}{2} = n\pi - \frac{\pi}{2}$$

~~If $4\theta = 0$,~~

Principle value:

$$\phi' - \phi = \cancel{\pi} - \pi/2 = \pi/2$$

$\therefore$ angle b/w 2 curves is $\pi/2$.

8\. Show that the circle $r = b$ cuts the curve $r^2 = a^2 \cos 2\theta + b^2$ at an angle $\tan^{-1}\left(\frac{a^2}{b^2}\right)$

Finding point of intersection.

$C1: r = b$

$C2: r^2 = a^2 \cos 2\theta + b^2$

Eliminating r.

$r^2 = b^2$ & $r^2 = a^2 \cos 2\theta + b^2$

$\cancel{b^2} = a^2 \cos 2\theta + \cancel{b^2}$

$\cos 2\theta = 0$

$2\theta = \pi/2 \Rightarrow \theta = \pi/4$

Differentiating ~~eq~~. C1.

$$\frac{dr}{d\theta} = 0$$

$$\tan\phi = \frac{r}{dr/d\theta} = \frac{b}{dr/d\theta} \xrightarrow{\text{ap.}} \infty$$

$\therefore \phi = \pi/2 \longrightarrow (1)$

Differentiating e2.

$\cancel{2r\frac{dr}{d\theta}}$

$\cancel{2}r\frac{dr}{d\theta} = -a^2 \cdot \cancel{2} \sin 2\theta$

$\frac{dr}{d\theta} = \frac{-a^2 \sin 2\theta}{r}$

$\tan\phi' = \frac{r}{dr/d\theta} = \frac{-r^2}{a^2 \sin 2\theta}$

$\tan\phi' = \frac{a^2\cos 2\theta + b^2}{-a^2 \sin 2\theta}$

$\tan\phi' = -\cot 2\theta + \frac{-b^2}{a^2 \sin 2\theta}$

$= \cot \pi/2 \; \frac{\phi - b^2}{a^2 \sin \pi/2}$

$\tan\phi' = 0 + \frac{-b^2}{a^2 \times 1}$

$-\tan\phi' = \frac{b^2}{a^2} = \tan(\pi - \phi')$

$\cancel{\pi - \phi' = \tan^{-1}\left(\frac{b^2}{a^2}\right)}$

$\cancel{\phi' = \pi - \tan^{-1}\left(\frac{b^2}{a^2}\right)}$

$\cot(\pi - \phi') = \frac{a^2}{b^2} = \tan^{-1}$

$\star$ $\tan\phi_1 \tan\phi_2 = -1 \Rightarrow$ curves $\perp$

20 $\because \tan(\phi_1-\phi_2) = \dfrac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}$

$$\tan^{-1}\left(\frac{a^2}{b^2}\right) = \phi' - \frac{\pi}{2}$$

$$\phi' = \frac{\pi}{2} + \tan^{-1}\left(\frac{a^2}{b^2}\right) \longrightarrow (2)$$

$$\phi' - \phi = \frac{\pi}{2} + \tan^{-1}\left(\frac{a^2}{b^2}\right) - \frac{\pi}{2}$$

$$\boxed{\therefore \phi' - \phi = \tan^{-1}\left(\frac{a^2}{b^2}\right)}$$

9. Show that the curves

$r = a\ln\theta$ (C1) and $r = \dfrac{a}{\ln\theta}$ (C2)

cut ~~orthogonally~~ at angle $2\tan^{-1}e$.

Eliminating r,

$$a\ln\theta = \frac{a}{\ln\theta}$$

$$(\ln\theta)^2 = 1$$

$\ln\theta = 1$ $\qquad$ $\ln\theta = -1$

$\theta = e$ $\qquad$ $\theta = \dfrac{1}{e}$ ← ignore this

Differentiating C1.

$$\frac{dr}{d\theta} = \frac{a}{\theta} \Rightarrow \tan\phi = \frac{r}{dr/d\theta}$$

$$\tan\phi = \frac{r\theta}{a} = \frac{a\ln\theta\,\theta}{a} = \theta\ln\theta$$

10.

Differentiating c2.

$$\frac{dr}{d\theta} = \frac{-a}{(\ln\theta)^2}\cdot\frac{1}{\theta}$$

$$\tan\phi' = \frac{r}{dr/d\theta} = \frac{-r\theta(\ln\theta)^2}{a} = \frac{-\cancel{a}}{\cancel{\ln\theta}}\cdot\frac{\theta(\ln\theta)^{\cancel{2}}}{\cancel{a}}$$
$$= -\theta\ln\theta$$

$\theta = e \Rightarrow \tan\phi = e\ln e = e.$

$\tan\phi' = -\theta\ln\theta = -e\ln e = -e.$

$\phi = \tan^{-1}e \quad \phi' = \tan^{-1}(-e)$
$= -\tan^{-1}(e)$

$\phi - \phi' = \tan^{-1}e + \tan^{-1}e$
$= 2\tan^{-1}e.$

If $\theta = 1/e \Rightarrow \tan\phi = -1/e$ & $\tan\phi' = 1/e.$

$\phi = -\tan^{-1}(1/e)$ & $\phi' = \tan^{-1}(1/e)$

$\phi' - \phi = 2\tan^{-1}(1/e).$

10. Find the angle of intersection of the curves $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Eliminating r.

$$\frac{\cancel{a}\theta}{1+\theta} = \frac{\cancel{a}}{1+\theta^2}$$

$\theta(1+\theta^2) = 1+\theta.$

$\theta^3 + \cancel{\theta} = 1 + \cancel{\theta}$

$\theta^3 = 1$

Taking real values

$$\boxed{\theta = 1}$$

Diff. C1.

$$\frac{dr}{d\theta} = a\left(\frac{d}{d\theta}\left(\frac{\theta+1-1}{1+\theta}\right)\right)$$

$$\frac{dr}{d\theta} = a\frac{d}{d\theta}\left(\frac{\theta \cancel{+1}}{1+\theta}\right)$$

$$\frac{dr}{d\theta} = a(1)$$

$$\tan\phi = \frac{r}{dr/d\theta} = \frac{r}{a} = \frac{\cancel{a}\theta}{(1+\theta)(a)}$$

$$\tan\phi = \theta+1 = 1+1 = 2$$

Diff. C2

$$\frac{dr}{d\theta} = a\frac{d}{d\theta}\left((1+\theta^2)^{-1}\right)$$

$$\frac{dr}{d\theta} = \frac{a(-1)}{(1+\theta^2)^2} \cdot 2\theta$$

$$\frac{dr}{d\theta} = \frac{-2a\theta}{(1+\theta^2)^2}$$

$$\tan\phi' = \frac{-r(1+\theta^2)^2}{2a\theta} = \frac{-\cancel{a}(1+\theta^2)^{\cancel{2}}}{\cancel{(1+\theta^2)}\,2\cancel{a}\theta}$$

$$\tan\phi' = \frac{-(1+\theta^2)}{2\theta}$$

~~$\phi = \tan^{-1} 2$~~

Diff. C1.

$$\frac{dr}{d\theta} = a \frac{d}{d\theta}\left(\frac{\theta}{1+\theta}\right)$$

$$= a\left(\frac{(1)(1+\theta) \cancel{\theta(1+\theta)} - \theta(1)}{(1+\theta)^2}\right)$$

$$\frac{dr}{d\theta} = a\left(\frac{1+\cancel{\theta}-\cancel{\theta}}{(1+\theta)^2}\right) = \frac{a}{(1+\theta)^2}$$

$$\tan\phi = \frac{r(1+\theta)^2}{a} = \frac{\cancel{a}\theta}{\cancel{(1+\theta)}\,\cancel{a}}(1+\theta)^{\cancel{2}}$$

$$\tan\phi = \theta(1+\theta) = 1(2) = 2$$

$$\tan\phi' = \frac{-(1+1)}{2} = -1$$

$$\phi = \tan^{-1} 2 \qquad \phi' = \tan^{-1}(-1) \; \cancel{\tan^{-1}}$$

$$= \tan^{-1} 2 - \tan^{-1}(-1) = \phi - \phi'$$
$$= \tan^{-1} 2 + \tan^{-1} 1$$
$$= \tan^{-1}\left(\frac{2+1}{1-2}\right) = \tan^{-1}(-3)$$

$$\therefore \phi' - \phi = \tan^{-1} 3.$$

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11. Show that the two curves $r^2 \sin 2\theta = a^2$ and $r^2 \cos 2\theta = b^2$ cut orthogonally.

$$\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta} = \frac{a^2}{b^2}$$

$$\tan 2\theta = \frac{a^2}{b^2} \Rightarrow 2\theta = \tan^{-1}\left(\frac{a^2}{b^2}\right)$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{a^2}{b^2}\right)$$

Consider $r^2 \sin 2\theta = a^2$

$$2 \ln r + \ln \sin 2\theta = \ln a^2$$

Diff.

$$\frac{\cancel{2}}{r}\frac{dr}{d\theta} + \cancel{2}\cdot\frac{\cos 2\theta}{\sin 2\theta} = 0$$

$$\cot \phi = -\cot 2\theta$$

$$\tan \phi = -\tan 2\theta \rightarrow (1)$$

Consider $r^2 \cos 2\theta = b^2$

$$2 \ln r + \ln \cos 2\theta = \ln b^2$$

Diff.

$$\frac{\cancel{2}}{r}\frac{dr}{d\theta} + \cancel{(2)}\frac{-\sin 2\theta}{\cos 2\theta} = 0$$

$$\cot \phi' = \tan 2\theta.$$

$$\tan \phi' = \cot 2\theta \longrightarrow (2)$$

Multiplying (1) & (2).

$$\tan\phi \tan\phi' = -\tan 2\theta \cot 2\theta \quad [\theta \text{ are same}]$$

$$\boxed{\tan\phi \tan\phi' = -1}$$

Hence the curves are orthogonal

12. Find the p-r equation (pedal equation) of
$r^m \cos m\theta = a^m \rightarrow (1)$

Taking ln on both sides

$m \ln r + \ln \cos m\theta = \ln a^m$

Differentiating

$$\frac{\cancel{m}}{r}\frac{dr}{d\theta} + \frac{-\sin m\theta}{\cos m\theta}\cancel{(m)} = 0$$

$$\cot \phi = \tan m\theta = \cot\left(\frac{\pi}{2} - m\theta\right)$$

$$\phi = \frac{\pi}{2} - m\theta$$

The pedal equation is

$$\boxed{p = r\sin\phi} = r\sin\left(\frac{\pi}{2} - m\theta\right)$$

$$p = r\cos m\theta \rightarrow (2)$$

From eq. 1.

$$\cancel{\frac{r^m}{a^m}} \quad \frac{a^m}{r^m} = \cos m\theta$$

$\therefore$ (2) becomes

$$p = r\frac{a^m}{r^m} = \frac{a^m}{r^{m-1}}$$

$$\boxed{p r^{m-1} = a^m}$$

Hyperbolic functions.

$$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2} \; ; \; \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} \; ; \; \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\cosh^2\theta - \sinh^2\theta = 1 \; ; \; \cos^2\theta + \sin^2\theta = 1$$

$$1 - \operatorname{sech}^2\theta = \tanh^2\theta \; ; \; 1 + \tan^2\theta = \sec^2\theta$$

$$\coth^2\theta - 1 = \operatorname{cosech}^2\theta \; ; \; 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Derivatives

$$\frac{d}{d\theta}(\sinh\theta) = \cosh\theta$$

$$\frac{d}{d\theta}(\cosh\theta) = \sinh\theta$$

$$\frac{d}{d\theta}(\tanh\theta) = \operatorname{sech}^2\theta$$

$$\frac{d}{d\theta}(\operatorname{sech}\theta) = -\operatorname{sech}\theta\tanh\theta$$

$$\frac{d}{d\theta}(\coth\theta) = -\operatorname{cosech}^2\theta$$

$$\frac{d}{d\theta}(\operatorname{cosech}\theta) = -\operatorname{cosech}\theta\coth\theta$$

13. Find the pedal equation of the curve $r = a\,\text{sech}(n\theta)$.

$$\frac{dr}{d\theta} = -an\,\text{sech}(n\theta)\tanh(n\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\cancel{a}n\,\cancel{\text{sech}(n\theta)}\tanh(n\theta)}{\cancel{a}\,\cancel{\text{sech}(n\theta)}}$$

$$\cot\phi = -n\tanh(n\theta)$$

$$\cot\phi = -n\tanh(n\theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\frac{1}{p^2} = \frac{1}{r^2}(1+n^2\tanh^2 n\theta)$$

$$= \frac{1}{r^2}(1+n^2(1-\text{sech}^2 n\theta))$$

$$\cancel{\frac{1}{p^2} = \frac{1}{r^2}(1+n^2-n^2\text{sech}^2 n\theta)}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1+n^2\left(1-\frac{r^2}{a^2}\right)\right)$$

$$= \frac{1}{r^2}\left(1+n^2-\frac{n^2r^2}{a^2}\right)$$

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2}+\frac{n^2}{r^2}-\frac{n^2}{a^2}}$$

14. Find the pedal equation of the curve

$$r^m = a^m \sin m\theta + b^m \cos m\theta$$

Diff. wrt θ

$$\cancel{m} \cdot r^{m-1} \frac{dr}{d\theta} = \cancel{m}\, a^m \cos m\theta - \cancel{m}\, b^m \sin m\theta$$

$$\frac{r^m}{r} \frac{dr}{d\theta} = a^m \cos m\theta - b^m \sin m\theta.$$

$$r^m \cot\phi = a^m \cos m\theta - b^m \sin m\theta$$

$$\cot\phi = \frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta} \rightarrow (1)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \cot^2\phi\right)$$

(1) (dividing by $\cos m\theta$)

$$\cot\phi = \frac{a^m - b^m \tan m\theta}{a^m \tan m\theta + b^m}$$

$$= \frac{\frac{a^m}{b^m} - \tan m\theta}{1 + \frac{a^m}{b^m} \tan m\theta}$$

$= \cancel{\tan}$

Let $\frac{a^m}{b^m} = \tan\omega$

$$\cot\phi = \frac{\tan\omega - \tan m\theta}{1+\tan\omega\tan\theta m)}$$

$$\cot\phi = \tan(\omega - m\theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{(a^m\cos m\theta - b^m\sin m\theta)^2}{(a^m\sin m\theta + b^m\cos m\theta)^2}\right)$$

$$= \frac{1}{r^2}\left(\frac{(a^m\sin m\theta + b^m\cos m\theta)^2 + (a^m\cos m\theta - b^m\sin m\theta)^2}{(a^m\sin m\theta + b^m\cos m\theta)^2}\right)$$

$$= \frac{1}{r^2}\left(\frac{a^{2m}\sin^2 m\theta + b^{2m}\cos^2 m\theta + a^{2m}\cos^2 m\theta + b^{2m}\sin^2 m\theta}{(a^m\sin m\theta + b^m\cos m\theta)^2}\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(\frac{a^{2m} + b^{2m}}{r^{2m}}\right)$$

$$\boxed{r^{2m+2} = p^2(a^{2m} + b^{2m})}$$

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15. Find the pedal equation of $\dfrac{r}{\sqrt{a^2-b^2}} = \sec\left[\left(\dfrac{\sqrt{a^2-b^2}}{a}\right)\theta\right]$

Let $c = \sqrt{a^2-b^2}$

$$\therefore \quad \frac{r}{c} = \sec\left[\left(\frac{c}{a}\right)\theta\right]$$

Taking derivative.

$$\frac{1}{c}\frac{dr}{d\theta} = \frac{c}{a}\sec\left(\frac{c\theta}{a}\right)\tan\left(\frac{c}{a}\theta\right)$$

$$\frac{dr}{d\theta} = \frac{c^2}{a} \sec\left(\frac{c\theta}{a}\right) \tan\left(\frac{c\theta}{a}\right)$$

$$r = c \sec\left(\frac{c\theta}{a}\right)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{c^{\cancel{2}} \cancel{\sec\left(\frac{c\theta}{a}\right)} \tan\left(\frac{c\theta}{a}\right)}{a \cancel{c} \cancel{\sec\left(\frac{c\theta}{a}\right)}}$$

$$\cot\phi = \frac{c}{a} \tan\left(\frac{c\theta}{a}\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \cot^2\phi\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{c^2}{a^2} \tan^2\left(\frac{c\theta}{a}\right)\right)$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{c^2}{a^2}\left(\sec^2\left(\frac{c\theta}{a}\right) - 1\right)\right)$$

$$= \frac{1}{r^2}\left(1 + \frac{c^2}{a^2} \sec^2\left(\frac{c\theta}{a}\right) - \frac{c^2}{a^2}\right)$$

$$= \frac{1}{r^2}\left(1 + \frac{r^2}{a^2} - \frac{c^2}{a^2}\right)$$

$$= \frac{1}{r^2} + \frac{1}{a^2} - \frac{c^2}{a^2 r^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{a^2} - \frac{(a^2-b^2)}{a^2r^2}$$

$$= \cancel{\frac{1}{r^2}} + \frac{1}{a^2} - \cancel{\frac{1}{r^2}} + \frac{b^2}{a^2r^2}$$

$$\frac{1}{p^2} = \frac{r^2}{a^2r^2} + \frac{b^2}{a^2r^2}$$

$$\frac{a^2r^2}{p^2} = r^2+b^2$$

$$\boxed{a^2r^2 = p^2(r^2+b^2)}$$

Pedal equation in cartesian form

$$ax + by + c = 0$$

$$y - y_1 = m(x-x_1)$$

$$m = \frac{dy}{dx}$$

$$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right| = p.$$

From origin,

$$p = \frac{|c|}{\sqrt{a^2+b^2}}$$

16. Find the pedal equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating wrt x.

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{x}{a^2} = \frac{y}{b^2}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{xb^2}{ya^2}$$

Equation of tangent through (x, y):

$$m = \frac{x}{y}\frac{b^2}{a^2}$$

$$(Y - y) = \frac{xb^2}{ya^2}(X - x)$$

$$ya^2(Y - y) = xb^2(X - x)$$

$$Ya^2y - y^2a^2 = Xxb^2 - x^2b^2$$

tangent: $(xb^2)X - (a^2y)Y + y^2a^2 - x^2b^2 = 0$

$\perp$ distance of tangent from origin

$$p = \left| \frac{(xb^2)(0) - (a^2y)(0) + y^2a^2 - x^2b^2}{\sqrt{x^2b^4 + a^4y^2}} \right|$$

$$p = \frac{|y^2a^2 - x^2b^2|}{\sqrt{x^2b^4 + a^4y^2}}$$

$$\cancel{p^2 = \frac{(y^2a^2 - x^2b^2)^2}{x^2b^4 + a^4y^2}}$$

$$\cancel{p^2 =}$$

$$(xb^2)X - (a^2y)Y + a^2y^2 - x^2b^2 = 0$$

Dividing by a^2b^2

$$\frac{x}{a^2}X - \frac{y}{b^2}Y + \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 0$$

From given eq.

$$\boxed{\frac{x}{a^2}X - \frac{y}{b^2}Y - 1 = 0}$$

equation of tangent

Length of perpendicular from $(0,0)$ to tangent

$$p = \frac{\left|\frac{x}{a^2}(0) + -\frac{y}{b^2}(0) - 1\right|}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}}$$

$$p = \left|\frac{-1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}}\right|$$

$$p^2 = \frac{1}{\frac{x^2}{a^4} + \frac{y^2}{b^4}}$$

$$\cancel{\frac{1}{p^2} = \frac{a^4}{x^2} + 1}$$

$$\frac{1}{p^2} = \left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2 \longrightarrow (1)$$

$$r^2 = x^2 + y^2$$

substituting $y^2 = r^2 - x^2$ in given eq.

$$\cancel{\frac{1}{P}} = \frac{x^2}{a^2} - \frac{(r^2 - x^2)}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{r^2}{b^2} + \frac{x^2}{b^2} = 1$$

$$x^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{b^2 + r^2}{b^2}$$

$$x^2 = \frac{\frac{b^2+r^2}{b^2}}{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{\frac{b^2+r^2}{\cancel{b^2}}}{\frac{a^2+b^2}{a^2\cancel{b^2}}}$$

$$x^2 = \frac{(r^2+b^2)(a^2)}{(a^2+b^2)}$$

$$\boxed{\frac{x^2}{a^2} = \frac{r^2+b^2}{a^2+b^2}}$$

$$\therefore \frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 \quad \text{(from given eq.)}$$

$$\frac{y^2}{b^2} = \frac{r^2 \cancel{+b^2} - a^2 \cancel{-b^2}}{a^2+b^2} = \frac{r^2-a^2}{a^2+b^2}$$

$$\boxed{\frac{y^2}{b^2} = \frac{r^2 - a^2}{a^2+b^2}}$$

Substituting for $\frac{x^2}{a^2}$ & $\frac{y^2}{b^2}$ in eq. (1).

$$\frac{1}{p^2} = \left(\frac{x^2}{a^2}\right)\frac{1}{a^2} + \left(\frac{y^2}{b^2}\right)\frac{1}{b^2}$$

$$\boxed{\frac{1}{p^2} = \left(\frac{r^2+b^2}{a^2+b^2}\right)\frac{1}{a^2} + \left(\frac{r^2-a^2}{a^2+b^2}\right)\frac{1}{b^2}}$$

$$\frac{1}{p^2} = \frac{b^2(r^2+b^2) + a^2(r^2-a^2)}{a^2b^2(a^2+b^2)}$$

$$\frac{1}{p^2} = \frac{r^2(b^2+a^2) + b^4 - a^4}{a^2b^2(a^2+b^2)}$$

$$\frac{1}{p^2} = \frac{r^2}{a^2b^2} + \frac{(b^2-a^2)}{a^2b^2}$$

$$\frac{a^2b^2}{p^2} = r^2 + b^2 - a^2$$

$$a^2b^2 = p^2(r^2 + b^2 - a^2)$$

17. Find the pedal equation of the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiating, wrt x,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-xb^2}{a^2y}$$

tangent at (x,y)

$$(X-x)\frac{dy}{dx}=(Y-y)$$

$$(X-x)\left(\frac{-xb^2}{a^2y}\right)=(Y-y)$$

$$X\left(\frac{-xb^2}{a^2y}\right)+\frac{x^2b^2}{a^2y}=Y-y$$

$$X\left(\frac{-x}{a^2}\right)+\frac{x^2}{a^2}=(Y-y)\frac{y}{b^2}$$

$$-x\left(\frac{X}{a^2}\right)+\frac{x^2}{a^2}=Y\left(\frac{y}{b^2}\right)-\frac{y^2}{b^2}$$

$$X\left(\frac{-x}{a^2}\right)+1=Y\left(\frac{y}{b^2}\right)$$

$$\boxed{X\left(\frac{x}{a^2}\right)+Y\left(\frac{y}{b^2}\right)=1}$$ Tangent at (x,y)

~~At the point~~

Distance from (0,0) ~~from~~ to tangent

$$p = \frac{|-1|}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}}$$

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4}$$

Pedal eq. is p-r eq.

$$r^2 = x^2 + y^2$$

$$y^2 = r^2 - x^2$$

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{r^2 - x^2}{b^4}$$

$$\frac{1}{p^2} = x^2\left(\frac{1}{a^4} - \frac{1}{b^4}\right) + \frac{r^2}{b^4}$$

$$\frac{x}{a^2} + \frac{(r^2 - x^2)}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{x^2}{b^2} = \frac{b^2 - r^2}{b^2}$$

$$x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{b^2 - r^2}{b^2}$$

$$\cancel{\frac{1}{p^2}}\, x^2 = \left(\frac{b^2 - r^2}{\cancel{b^2}}\right)\left(\frac{a^2 \cancel{b^2}}{b^2 - a^2}\right)$$

$$\boxed{\frac{x^2}{a^2} = \frac{b^2 - r^2}{b^2 - a^2}}$$

$$\frac{y^2}{b^2} = \frac{\cancel{b^2} - a^2 - \cancel{b^2} + r^2}{b^2 - a^2} = \frac{r^2 - a^2}{b^2 - a^2}$$

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4}$$

$$\frac{1}{p^2} = \frac{1}{a^2}\left(\frac{b^2 - r^2}{b^2 - a^2}\right) + \frac{1}{b^2}\left(\frac{r^2 - a^2}{b^2 - a^2}\right)$$

$$\frac{a^2 b^2}{p^2} = \frac{b^2(b^2 - r^2) + a^2(r^2 - a^2)}{b^2 - a^2}$$

$$\frac{a^2 b^2}{p^2} = \frac{r^2(a^2 - b^2) + b^4 - a^4}{(-1)(a^2 - b^2)}$$

$$= \frac{(a^2 - b^2)(r^2) + (b^2 - a^2)(b^2 + a^2)}{(-1)(a^2 - b^2)}$$

$$= \frac{r^2 - b^2 - a^2}{-1}$$

$$\frac{a^2 b^2}{p^2} = a^2 + b^2 - r^2$$

$$\boxed{p^2(a^2 + b^2 - r^2) = a^2 b^2}$$

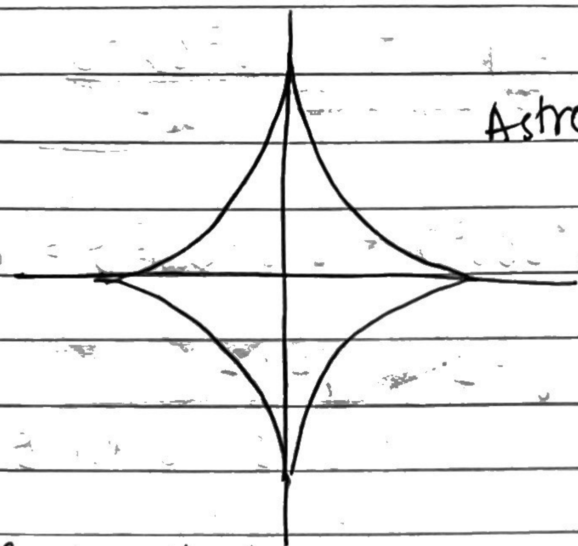
$$x^2 + y^2 = r^2$$
$$x = r\cos\theta , \quad y = r\sin\theta .$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1.$$

$$r^2b^2\cos^2\theta + r^2a^2\sin^2\theta = a^2b^2$$

$$\boxed{r^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}}$$ ← polar form

18. Find the pedal equation of $x = a\cos^3 t$, $y = a\sin^3 t$.



Astroid curve
$x^{2/3} + y^{2/3} = a^{2/3}$

Diff. y wrt t

$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

Diff. x wrt t

$$\frac{dx}{dt} = -3a\cos^2 t\sin t$$

$$\frac{dy}{dx} = \frac{-3\cancel{a}\sin^2 t\,\cancel{\cos t}}{3\cancel{a}\cos^2 t\,\cancel{\sin t}} = -\tan t.$$

$$m = -\tan t.$$

equation of tangent at (x, y)

$$Y - y = (-\tan t)(X - x)$$

$$Y + \tan t\, X = y - x\tan t$$

Equation of tangent at $(a\cos^3 t, a\sin^3 t)$

$$y - a\sin^3 t = -\tan t\,(x - a\cos^3 t)$$

$$y + \tan t\, x \quad - a\sin^3 t - \tan t\, a\cos^3 t.$$

pedal distance from origin

$$p = \frac{|-a\sin^3 t - \tan t\, a\cos^3 t|}{\sqrt{1+\tan^2 t}}$$

$$p = \frac{a\sin^3 t + a\cos^3 t\tan t}{\sec t}$$

$$= (a\sin^3 t + a\cos^2 t\sin t)\cos t$$

$$= a\sin t(\sin^2 t + \cos^2 t)\cos t$$

$$p^2 = a^2\sin^2 t\cos^2 t \longrightarrow (1)$$

We know $r^2 = x^2 + y^2$

$$= a^2\cos^6 t + a^2\sin^6 t$$

$$= a^2(\cos^2 t + \sin^2 t)(\cos^4 t \cancel{+} \cos^2 t\sin^2 t + \sin^4 t)$$

$$r^2 = a^2(\cos^4 t + \sin^4 t \cancel{+} \cos^2 t\sin^2 t)$$

$$= a^2((\cos^2 t + \sin^2 t)^2 - 3\cos^2 t\sin^2 t)$$

$$= a^2(1 - 3\cos^2 t\sin^2 t)$$

$$r^2 = a^2 - 3a^2\cos^2 t\sin^2 t$$

From eq. (1).

$$\boxed{r^2 = a^2 - 3p^2}$$

pedal = distance from (0,0) to tang

19. Parametric version of hyperbola is given as:

$x = a\sec\theta \qquad y = b\tan\theta$

Use this to find the pedal equation of hyperbola

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta} = \frac{b\operatorname{cosec}\theta}{a}$$

slope of tangent = ~~(b cos θ~~ $(b\operatorname{cosec}\theta)/(a)$

Equation of tangent at $(a\sec\theta, b\tan\theta)$

$$(y - b\tan\theta) = (x - a\sec\theta)\left(\frac{b}{a}\operatorname{cosec}\theta\right)$$

$$ay - ab\tan\theta = b\operatorname{cosec}\theta\, x - ab\sec\theta\operatorname{cosec}\theta$$

$$0 = \underbrace{b\operatorname{cosec}\theta}_{A}\, x \underbrace{-\,a}_{B}y + \underbrace{ab\tan\theta - ab\sec\theta\operatorname{cosec}\theta}_{C}$$

Distance from (0,0) to the line is given by

$$p = \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}}$$

$$\gamma^2 = x^2 + y^2 = a^2\sec^2\theta + b^2\tan^2\theta$$
$$= a^2(1+\tan^2\theta) + b^2\tan^2\theta$$

$$\cancel{p = \frac{|C|}{\sqrt{A^2+B^2}} = \frac{ab(\tan\theta - \sec\theta\,\mathrm{cosec}\,\theta)}{\sqrt{\ }}}$$

$$\gamma^2 = a^2 + (b^2+a^2)\tan^2\theta \Rightarrow \boxed{\tan^2\theta = \frac{\gamma^2 - a^2}{a^2+b^2}}$$

$0 = (b\,\mathrm{cosec}\,\theta)x - ay + ab(\tan\theta - \sec\theta\,\mathrm{cosec}\,\theta)$

multiplying by $\sin\theta\cos\theta$

$$(b\cos\theta)x - a\sin\theta\cos\theta\,y + ab(\sin^2\theta - 1) = 0$$

$$(b\cos\theta)x - a\sin\theta\cos\theta\,y - ab\cos^2\theta = 0$$

$$p = \frac{|-ab\cos^2\theta|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta\cos^2\theta}}$$

$$p^2 = \frac{a^2b^2\cos^2\theta}{b^2 + a^2\sin^2\theta}$$ (Dividing by $\cos^2\theta$)

$$p^2 = \frac{a^2b^2}{b^2\sec^2\theta + a^2\tan^2\theta} = \frac{a^2b^2}{b^2 + (a^2+b^2)\tan^2\theta}$$

$$p^2 = \frac{a^2b^2}{b^2 + \gamma^2 - a^2}$$

$$\boxed{a^2b^2 = p^2(\gamma^2 + b^2 - a^2)}$$

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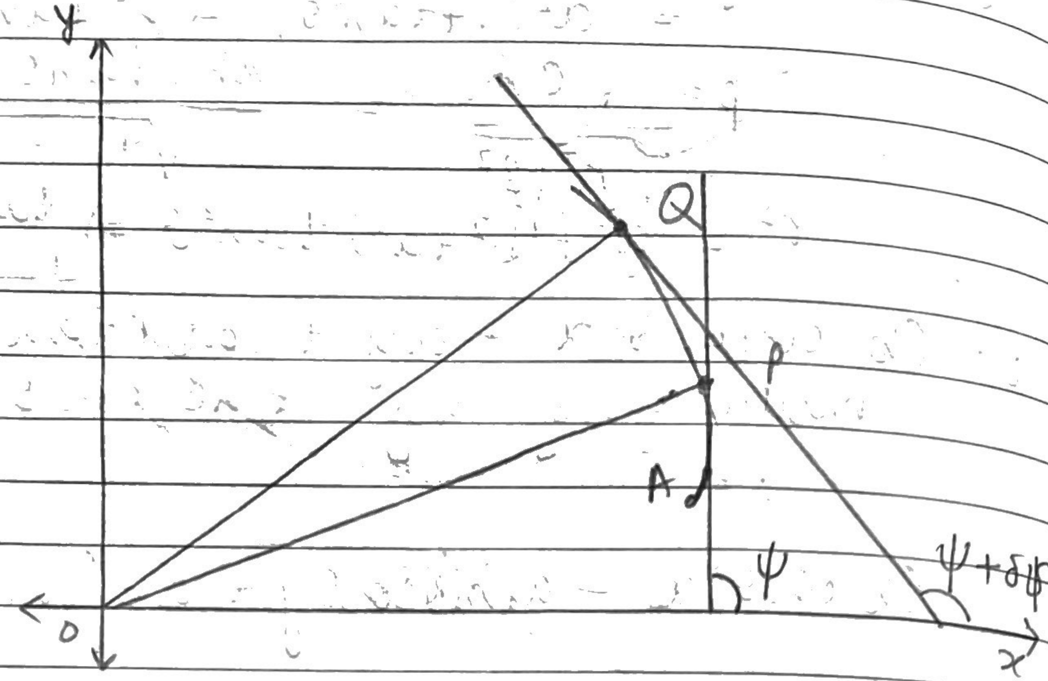
Radius of curvature



Let P & Q be two neighbouring points on the given curve.

Let A be a fixed point on the curve from which the arc distances are measured.

Let arc AP $= s$

arc AQ $= s + \delta s$

So that arc PQ $= \delta s$

Let the tangents at P and Q make angles ψ and $\psi + \delta\psi$ ~~&~~ respectively with the x-axis. Then, the angle between the tangents is $\delta\psi$

In moving from P to Q, ~~the~~ tangent has turned through an angle $\delta\psi$. This is called total bending or total curvature of arc PQ.

The ratio $\delta\psi/\delta s$ is called the average curvature of arc PQ.

In the limiting case, when the point Q approaches the point P, we have $\delta s \longrightarrow 0$.
The limit value of $\frac{\delta\psi}{ds}$ is called the curvature of the curve at P.

$$\text{Curvature at } P = \lim_{\substack{\delta \to 0 \\ (Q \to P)}} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds} = K \text{ (kappa)} \quad (\text{radians length}^{-1})$$

The reciprocal of curvature at P is called the radius of curvature at P, denoted by ρ (rho)

IMPORTANT

1. ρ in Cartesian form $y = f(x)$ * if $\frac{dy}{dx} \to \infty$, replace x & y and $\frac{dx}{dy} \to 0$.

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \qquad y_1 = \frac{dy}{dx}$$

2. ρ in Parametric form $x = x(t)$, $y = y(t)$.

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} \qquad x' = \frac{dx}{dt}$$

3. ρ in Polar form $r = f(\theta)$

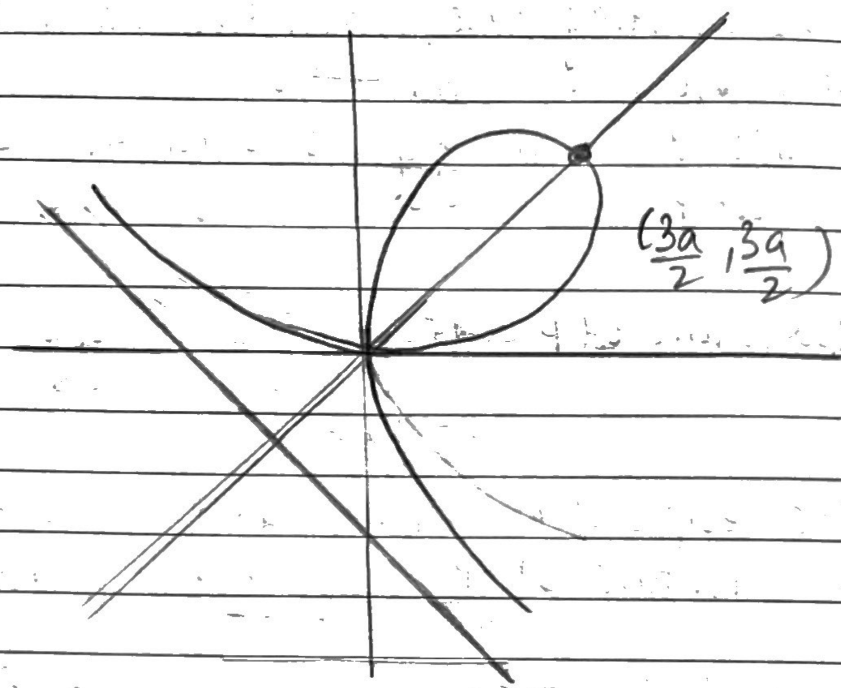
$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{(r^2 + 2r_1^2 - rr_2)} \qquad r_1 = \frac{dr}{d\theta}$$

4. ρ in Pedal form $p = f(r)$

$$\rho = r\frac{dr}{dp}$$

1. Find ρ of the curve $x^3+y^3=3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

(Folium of Descartes) crosses origin twice



Differentiating wrt x

$$\cancel{3}x^2 + \cancel{3}y^2\frac{dy}{dx} = \cancel{3}a\left(y + x\frac{dy}{dx}\right)$$

$$x^2 + y^2 y_1 = ay + axy_1$$

$$y_1(y^2-ax) = ay - x^2 \rightarrow (1)$$

$$\boxed{y_1 = \frac{ay-x^2}{y^2-ax}} \rightarrow (2)$$

Differentiating (1) wrt.

$$y_2(y^2-ax) + y_1(2yy_1 - a) = ay_1 - 2x$$

$$y_2(y^2-ax) = ay_1 - 2x - 2y(y_1)^2 + ay_1$$

$$\boxed{y_2 = \frac{2ay_1 - 2x - 2y(y_1)^2}{y^2 - ax}}$$

Value of y_1 at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

let $\frac{3a}{2} = b = x = y$.

$$y_1 = \frac{ab - b^2}{b^2 - ab} = -1$$

$$\boxed{y_1 = -1}$$

Value of y_2 at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ $x = y = b$.

$$y_2 = \frac{2a(-1) - 2b - 2b(-1)^2}{b^2 - ab}$$

$$= \frac{-2a - 2b - 2b}{b^2 - ab} = \frac{-2(a + 2b)}{b^2 - ab}.$$

$$y_2 = \frac{-2(a + 3a)}{\frac{9a^2}{4} - a \times \frac{3a \times 2}{2 \times 2}} = \frac{-2 \times 4a \times 4}{9a^2 - 6a^2}$$

$$y_2 = \frac{-32a}{3a^2} = \boxed{\frac{-32}{3a}}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{-32} \times 3a = -\frac{2\sqrt{2} \times 3a}{32}$$

$$\boxed{\rho = \frac{+3a\sqrt{2}}{16}}$$

2. Find ρ of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4},\frac{a}{4}\right)$

Differentiating wrt x.

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y_1=0$$

$$\boxed{y_1=\frac{-\sqrt{y}}{\sqrt{x}}} \longrightarrow (1)$$

At $(a/4, a/4)$, $\boxed{y_1=-1}$

Differentiating (1) wrt x

$$y_2=-\frac{\left[\left(\frac{1}{2\sqrt{y}}y_1\right)(\sqrt{x})-\frac{(\sqrt{y})(1)}{(2\sqrt{x})}\right]}{x}$$

$$y_2=-\left(\frac{y_1\sqrt{x}}{2\sqrt{y}}-\frac{\sqrt{y}}{2\sqrt{x}}\right)\times\frac{1}{x}$$

$$=-\left(\frac{(-1)}{2}-\frac{1}{2}\right)\times\frac{1}{x}$$

$$y_2=\frac{-1}{x}=\frac{-1}{a/4}=\boxed{\frac{-4}{a}=y_2}$$

$$\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\left|\frac{(1+1)^{3/2}a}{-4}\right|=\frac{2\sqrt{2}a}{4}$$

$$\boxed{\rho=\frac{a}{\sqrt{2}}}$$

3. Find ρ for the curve $x^2y = a(x^2+y^2)$ at $(-2a, 2a)$

Differentiating wrt x.

$$2xy + x^2\frac{dy}{dx} = a\left(2x + 2y\frac{dy}{dx}\right)$$

$$y_1(x^2 - 2ay) = 2ax - 2xy \rightarrow (1)$$

$$\boxed{y_1 = \frac{2ax - 2xy}{x^2 - 2ay}} \rightarrow (2)$$

Diff. (1) wrt x.

$$y_2(x^2 - 2ay) + y_1(2x - 2ay_1) = 2a - 2(y - xy_1)$$

$$y_2(x^2 - 2ay) + 2xy_1 - 2a(y_1)^2 = 2a - 2y + 2xy_1 \rightarrow (3)$$

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}$$

In (2) $(y = -x)$.

$$y_1 = \frac{2a\cancel{x} + 2x^{\cancel{2}}}{x^{\cancel{2}} + 2a\cancel{x}} = \frac{2(a+x)}{x+2a}$$

$$y_1 = \frac{2(a-2a)}{-2a+2a} = \frac{2(-a)}{0} \rightarrow \infty$$

$$\therefore x_1 = \frac{dx}{dy} = \frac{x^2 - 2ay}{2ax - 2xy} = 0.$$

$$\cancel{x_2 = \frac{d^2x}{dy^2} = (2x - 2a}$$

$$\frac{dx}{dy} = \frac{x^2 - 2ay}{2ax - 2xy}$$

$$x_2 = \frac{(2xx_1 - 2a)(2ax - 2xy) - (x^2 - 2ay)(2ax_1 - 2x_1y - 2x)}{(2ax - 2xy)^2}$$

$$x_2 = \frac{\cancel{2}(xx_1 - a)(2ax - 2xy) - \cancel{2}(x^2 - 2ay)(ax_1 - x_1y - x)}{\cancel{2}(ax - xy)\,2(ax - xy)}$$

$x = -y$; $x_1 = 0$

$$x_2 = \frac{(-a)(2a\cancel{x} + 2x^{\cancel{2}}) - (x^2 + 2ax)(-\cancel{x})}{\cancel{x}(a+x)\,2(x)(a+x)}$$

$$= \frac{(-a)(2a + 2x) - (x^2 + 2ax)}{(a+x)(2)(x)(a+x)}$$

$x = 2a$

$$x_2 = \frac{(-a)(2a - 4a) - 4a^2 + 4a}{(a+x)(a-2a)(2x)}$$

$$= \frac{(-a)(-2a) - 4a^2 + 4a}{(-a)(-4a)(a+x)}$$

$$= \frac{-2a^2 - 4a^2 + 4a}{(-a)(-4a)(-a)}$$

$$x_2 = (2a + 2a + 4)(a^2)(-4a)$$

$= \cancel{4a + 4}$

$= 4a + 4$

$$x_2 = \frac{(2ax-2xy)(2xx_1-2a)-(x^2-2ay)(2ax_1-2x_1y-2x)}{(2ax-2xy)^2}$$

$$= \frac{(-4a^2+8a^2)(-2a) - \cancel{(4a^2-4a^2)(2)(2a)}}{4(-2a^2+4a^2)^2}$$

$$= \frac{(4a^2)(-2a)}{4(2a^2)^2} = \frac{-2a^3}{4a^4} = \frac{-1}{2a}$$

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

$$= \frac{(1+0)^{3/2}\,2a}{-1} = -2a$$

$$\boxed{\rho = 2a}$$

4. Find ρ for the curve $r^2 = a^2\cos 2\theta$ at any point on it.

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{(r^2 + 2r_1^2 - rr_2)}$$

Diff

$$\cancel{2}r\frac{dr}{d\theta} = -\cancel{2}a^2\sin 2\theta$$

$$\boxed{r_1 = \frac{-a^2\sin 2\theta}{r}}$$

$$r_2 = \frac{2(-a^2\cos 2\theta)(r) - (-a^2\sin 2\theta)(r_1)}{r^2}$$

$$r_2 = \frac{-2a^2\cos 2\theta\, r + a^2\sin 2\theta\, r_1}{r^2}$$

$$\rho = \frac{\left(r^2 + \frac{a^4\sin^2 2\theta}{r^2}\right)^{3/2}}{r^2 + \frac{2a^4\sin^2 2\theta}{r^2} - \cancel{r}\frac{(-2a^2\cos 2\theta r + a^2\sin 2\theta r_1)}{r}}$$

$$\rho = \frac{(r^4 + a^4\sin^2 2\theta)}{r^3\left(\frac{r^4 + 2a^4\sin^2 2\theta}{r^2} + \frac{2a^2\cos 2\theta r - a^2\sin 2\theta r_1}{r}\right)}$$

$$\cancel{\phi} \quad \tan\phi = \frac{r}{dr/d\theta}$$

$$\tan\phi = -\frac{r^2}{a^2\sin 2\theta} = -\frac{\cancel{a^2}\cos 2\theta}{\cancel{a^2}\sin 2\theta}$$

$$\tan\phi = -\cot 2\theta = -\tan\left(\frac{\pi}{2}-2\theta\right)$$

$$\cancel{\cot\phi = -\tan 2\theta} = \tan\left(2\theta-\frac{\pi}{2}\right)$$

$$\cancel{\frac{1}{p^2} = \frac{1}{r^2}\left(1+\cot^2\phi\right)}$$

$$\cancel{\frac{1}{p^2} = \frac{1}{r^2}\left(1+\tan^2 2\theta\right)}$$

$$\phi = 2\theta - \frac{\pi}{2}$$

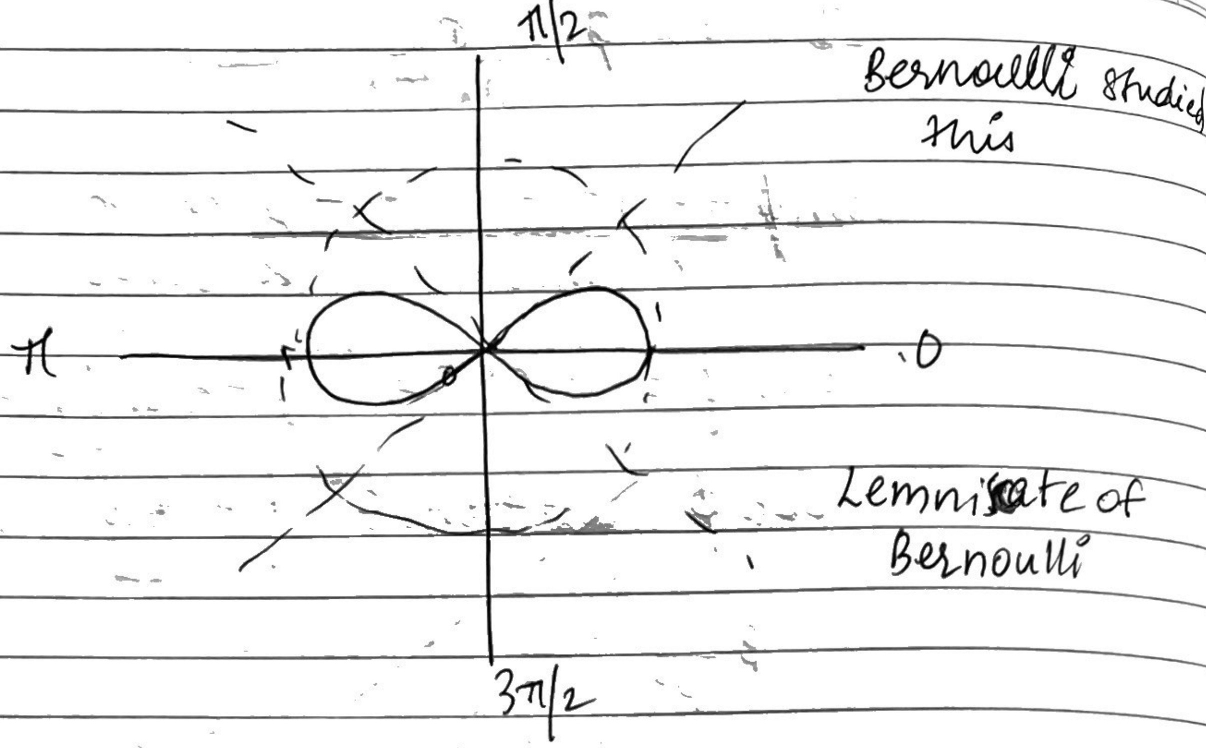
$$p = r\sin\phi = r\sin\left(2\theta-\frac{\pi}{2}\right)$$

$$p = -r\cos 2\theta$$

$$\boxed{p = \frac{-r^3}{a^2}}$$

$$1 = \frac{-3r^2}{a^2}\frac{dr}{dp}$$

$$\rho = r\cdot\frac{dr}{dp} = \frac{-ra^2}{3r^2} = \frac{-a^2}{3r}$$



r doesn't exist b/w $\theta = \pi/4$ & $\theta = 3\pi/4$

5. Find ρ for the curve $r = ae^{\theta\cot\alpha}$

$$\frac{dr}{d\theta} = a\cot\alpha\, e^{\theta\cot\alpha}$$

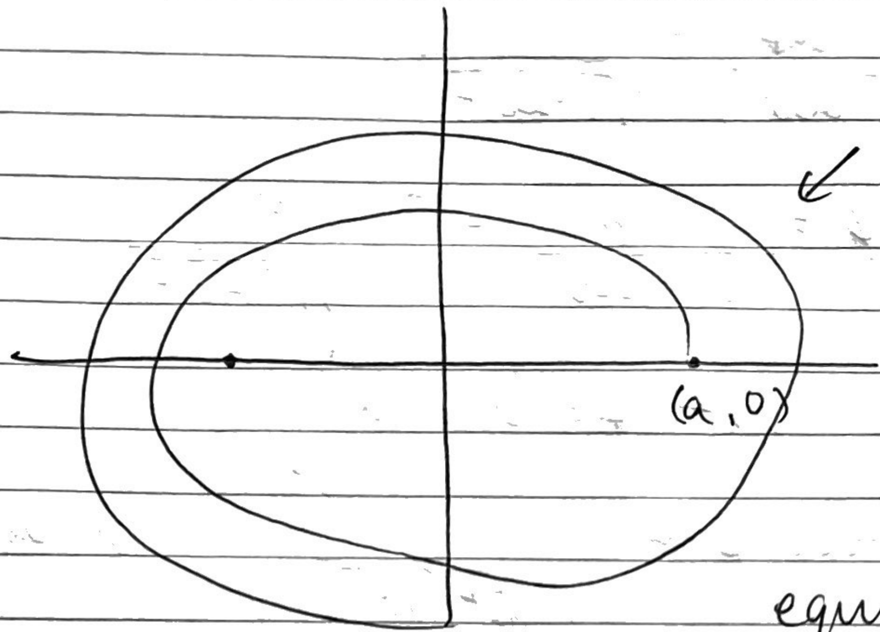
$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a}\cot\alpha\, \cancel{e^{\theta\cot\alpha}}}{\cancel{a}\,\cancel{e^{\theta\cot\alpha}}}$$

$$\cot\phi = \cot\alpha$$
$$\phi = \alpha$$
angle b/w tangent & radius is constant.

$$p = r\sin\phi = r\sin\alpha$$

$$1 = \sin\alpha \frac{dr}{dp}$$

$$\rho = r\frac{dr}{dp} = \boxed{\frac{r}{\sin\alpha} = \rho}$$



6. Find ρ for the curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$ at any point on it.

$$\theta = \frac{\sqrt{r^2-a^2}}{a} - \frac{\pi}{2} + \sin^{-1}\left(\frac{a}{r}\right)$$

Differentiating wrt r.

$$\frac{d\theta}{dr} = \frac{1}{a} \times \frac{(\cancel{2}r)}{\cancel{2}\sqrt{r^2-a^2}} + \frac{(a)}{\sqrt{1-\frac{a^2}{r^2}}} \cdot \left(\frac{-1}{r^2}\right)$$

$$\frac{d\theta}{dr} = \frac{r}{a\sqrt{r^2-a^2}} \quad \frac{-ar}{r^2\sqrt{r^2-a^2}}$$

$$= \frac{r^2}{ra\sqrt{r^2-a^2}} - \frac{a^2}{ar(\sqrt{r^2-a^2}}$$

$$= \frac{(r^2-a^2)}{ar\sqrt{r^2-a^2}} = \frac{\sqrt{r^2-a^2}}{ar}$$

$$\boxed{\frac{d\theta}{dr} = \frac{\sqrt{r^2-a^2}}{ar}}$$

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$$\frac{dr}{d\theta} = \frac{ar}{\sqrt{r^2-a^2}}$$

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{a}{\sqrt{r^2-a^2}}$$

let $r = a\sec\alpha$

$$\cancel{\cot\phi = \frac{a}{\sqrt{a^2\tan^2\alpha}} = \cot\alpha}$$

$$\cancel{\phi = \alpha = r}$$

can use Δ

$$\cot\phi = \frac{a}{\sqrt{r^2-a^2}}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1+\cot^2\phi\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1+\frac{a^2}{r^2-a^2}\right)$$

$$\frac{1}{p^2} = \frac{1}{\cancel{r^2}}\left(\frac{\cancel{r^2}\cancel{-a^2+a^2}}{r^2-a^2}\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2-a^2}$$

$$\boxed{p^2 = r^2-a^2}$$

Diff. wrt r.

$$\cancel{2}p\frac{dp}{dr} = \cancel{2}r$$

$$\boxed{\frac{dp}{dr} = \frac{r}{p}}$$

$$\rho = \frac{r\,dr}{dp} = \frac{r\,\cancel{p}}{r} = p$$

$$\boxed{\rho = p = \sqrt{r^2 - a^2}}$$

Find ρ for the curve $x = 2t^2 - t^4$, $y = 4t^3$ at the point $t = 1$.

$$\frac{dx}{dt} = 4t - 4t^3; \quad \text{at } t=1 \quad \cancel{\text{at } t} = 0 = x'$$

$$\frac{dy}{dt} = 12t^2; \quad \text{at } t = 1, \quad 12 = y'$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{(-y'x'' + x'y'')}$$

$$\frac{d^2x}{dt^2} = 4 - 12t^2; \quad x'' = -8$$

$$\frac{d^2y}{dt^2} = 24t; \quad y'' = 24$$

$$\rho = \frac{(+12 \times 12)^{3/2}}{(-12 \times -8)} = \frac{\overset{6}{\cancel{12}} \times \overset{3}{\cancel{12}} \times \cancel{12}}{\cancel{12} \times \cancel{8}}$$

$$\boxed{\rho = 18}$$

8. Find ρ for $x = t - \sin t$, $y = 1 - \cos t$ at $t = \pi$.

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$\frac{dx}{dt} = 1 - \cos t \qquad \frac{dy}{dt} = \sin t$$

$$\frac{d^2x}{dt^2} = \sin t \qquad \frac{d^2y}{dx^2} = \cos t$$

$$\rho = \frac{\left((1-\cos t)^2 + (\sin t)^2\right)^{3/2}}{(\cos t)(1-\cos t) - (\sin t)(\sin t)}$$

$$= \frac{(1 + 1 - 2\cos t)^{3/2}}{\cos t - 1}$$

$$= \frac{2^{3/2}(1-\cos t)^{3/2}}{(\cos t - 1)}$$

$$\rho = (-1)(2)^{3/2}(1-\cos t)^{1/2}$$

$$= (-1)(2)^{3/2}(1+1)^{1/2}$$

$$= (-1)\,4$$

$$\boxed{\rho = 4}$$

9. Find the ρ for the curve $x = a\ln\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)$

$y = a\sec\theta$.

$$\cancel{\frac{dx}{d\theta} = \frac{a\sec^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)} \times \frac{1}{2}}$$

$$\cancel{x = a\ln\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)}$$

$$\cancel{x = a\ln\left(\sin(\pi/4\right.}$$

$$\frac{dx}{d\theta} = \frac{a\sec^2(\pi/4+\theta/2)}{2\tan(\pi/4+\theta/2)} = \frac{a\sec(\pi/4+\theta/2)}{2\sin(\pi/4+\theta/2)}$$

$$= \frac{a}{2\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)} = \frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)$$

$$\boxed{\frac{dx}{d\theta} = \frac{a}{2}\sec(\theta) = x'}$$

$$\boxed{x'' = \frac{a}{2}\sec\theta\tan\theta}$$

$y' = a\sec\theta\tan\theta$.

$$y'' = a(\sec\theta\tan^2\theta + \sec^3\theta)$$
$$= a\sec\theta(\tan^2\theta + \sec^2\theta)$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{(x'y'' - x''y')}$$

$$= \frac{\left(a^2 \sec^2\theta + a^2 \sec^2\theta \tan^2\theta\right)^{3/2}}{a^2 \sec^2\theta (\tan^2\theta + \sec^2\theta) - a^2 \sec^2\theta \tan^2\theta}$$

$$= \frac{(a^2 \sec^2\theta)^{3/2} \left(1 + \tan^2\theta\right)^{3/2}}{a^2 \sec^2\theta \left(\sec^2\theta\right)}$$

$$= \frac{a^3 \sec^3\theta}{\cancel{a^2} \cancel{\sec^2\theta}} \; \frac{(1 + \tan^2\theta)^{3/2}}{\sec^2\theta}$$

$$= \frac{a \cancel{\sec\theta} \; \sec^{\cancel{3}\,2}\theta}{\cancel{\sec^2\theta}} = a\sec^2\theta$$

$$\boxed{\rho = a\sec^2\theta.}$$

10. Find ρ for the pedal curve $p = \frac{r^4}{r^2+a^2}$

Differentiating wrt r

$$\frac{dp}{dr} = \frac{4r^3(r^2+a^2) - r^4(2r)}{(r^2+a^2)^2}$$

$$\rho = r\frac{dr}{dp}$$

$$= \frac{\cancel{r}\,(r^2+a^2)^2}{4r^{\cancel{3}\,2}(r^2+a^2) - r^{\cancel{4}\,3}(2r)}$$

$$= \frac{(r^2+a^2)^2}{2r^2(2r^2+\cancel{2}a^2 - \cancel{2}r^2)}$$

$$\boxed{\rho = \frac{(r^2+a^2)^2}{2r^2(r^2+2a^2)}}$$

11. Find p-r equation of the rectangular hyperbola ($x^2-y^2=a^2$) $r^2\cos 2\theta = a^2$ and hence find ρ at any point

$$r^2 = a^2\sec 2\theta$$

Differentiating

$$\cancel{2}r\frac{dr}{d\theta} = \cancel{2}a^2\sec 2\theta\tan 2\theta$$

$$\frac{dr}{d\theta} = \frac{a^2\sec 2\theta.\tan\theta}{r}$$

$$\cot\phi = \frac{dr}{d\theta} \times \frac{1}{r} = \frac{\cancel{a^2 \sec 2\theta} \tan 2\theta}{\cancel{a^2 \sec 2\theta}}$$

$$\cot\phi = \tan 2\theta = \cot\left(\frac{\pi}{2} - 2\theta\right)$$

$$\phi = \frac{\pi}{2} - 2\theta$$

$$p = r\sin\phi = r\sin\left(\frac{\pi}{2} - 2\theta\right)$$

$$\boxed{p = r\cos 2\theta}$$

$$\sec 2\theta = r^2/a^2 \Rightarrow \cos 2\theta = a^2/r^2$$

$$p = \frac{ra^2}{r^2} = \frac{a^2}{r}$$

$$\boxed{p = \frac{a^2}{r}}$$

$$\rho = r\frac{dr}{dp}$$

$$1 = a^2 \frac{dr}{dp}\left(\frac{-1}{r^2}\right)$$

$$\frac{dr}{dp} = \frac{-r^2}{a^2}$$

$$\boxed{\rho = \frac{r^3}{a^2}}$$

12. ~~Find p-r eq. of ellipse~~ Find ρ of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Using p-r eq.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating wrt x

$$\frac{\cancel{2}x}{a^2} + \frac{\cancel{2}y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x b^2}{a^2 y}$$

Equation of tangent:

$$(Y-y) = (X-x)\left(\frac{-xb^2}{a^2y}\right)$$

$$Y-y = X\left(\frac{-xb^2}{a^2y}\right) + \frac{x^2b^2}{a^2y^2}$$

$$\frac{(Y-y)y}{b^2} = X\left(\frac{-x}{a^2}\right) + \frac{x^2}{a^2}$$

$$Y\left(\frac{y}{b^2}\right) = X\left(\frac{-x}{a^2}\right) + \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$Y\left(\frac{y}{b^2}\right) + X\left(\frac{x}{a^2}\right) = 1$$

Distance p from (0,0)

$$p = \left|\frac{-1}{\sqrt{\frac{y^2}{b^4} + \frac{x^2}{a^4}}}\right|$$

$$\frac{1}{p^2} = \frac{y^2}{b^4} + \frac{x^2}{a^4}.$$

$$r^2 = x^2 + y^2$$

$$y^2 = r^2 - x^2$$

$$\frac{x^2}{a^2} + \frac{r^2 - x^2}{b^2} = 1$$

$$x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{b^2 - r^2}{b^2}$$

$$x^2\left(\frac{b^2 - a^2}{a^2 \cancel{b^2}}\right) = \frac{(b^2 - r^2)}{\cancel{b^2}}$$

$$\frac{x^2}{a^2} = \frac{b^2 - r^2}{b^2 - a^2}$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$= \frac{\cancel{b^2} - a^2 - \cancel{b^2} + r^2}{b^2 - a^2}$$

$$\frac{y^2}{b^2} = \frac{r^2 - a^2}{b^2 - a^2}$$

$$\frac{1}{p^2} = \frac{b^2}{a^2b^4}\left(\frac{b^2-r^2}{b^2-a^2}\right) + \frac{a^2}{b^2a^4}\left(\frac{r^2-a^2}{b^2-a^2}\right)$$

$$\frac{1}{p^2} = \frac{1}{a^2b^2}\left(\frac{b^4-a^4+r^2(a^2-b^2)}{(b^2-a^2)}\right)$$

$$\frac{1}{p^2} = \frac{1}{a^2b^2}\left(b^2+a^2-r^2\right)$$

$$\boxed{a^2b^2 = p^2(a^2+b^2-r^2)}$$

Diff wrt p.

$$0 = \cancel{2}p\,(a^2+b^2-r^2) + p^2\left(-\cancel{2}r\frac{dr}{dp}\right)$$

$$\cancel{p}(a^2+b^2-r^2) = p^{\cancel{2}}r\frac{dr}{dp}$$

$$\rho = r\frac{dr}{dp} \qquad = \frac{a^2+b^2-r^2}{p}$$

$$\boxed{\rho = \frac{a^2b^2}{p^3}}$$

$$\rho = \frac{(a^2+b^2-r^2)^{3/2}}{ab}$$

26.08.19

Series Expansion

Taylor's Theorem (Generalised Mean Value theorem)

If a real-valued function $f(x)$ is such that

(i) f and its first $n-1$ derivatives are all continuous in $[a, a+h]$

(ii) $f^{(n)}(x)$ exists (n^{th} derivative) for all x in $(a, a+h)$

then there exists at least one number θ, $0<\theta<1$ such that

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

* $f^n \longrightarrow f(f(f \cdots (x))n$	
* $f^{(n)} \longrightarrow f'''''(x)$	

Where the $(n+1)^{th}$ term is called the Lagrange's Remainder denoted by R_n

Corollary:

1 When $n=1$, in Taylor's Theorem, we have

$$f(a+h) = f(a) + h f'(a+\theta h)$$

$$f'(a+\theta h) = \frac{f(a+h) - f(a)}{h}$$

$a < a+\theta h < a+h \longrightarrow c \in (a, a+h)$

$a+h \longrightarrow b$

$h \longrightarrow b-a$

Which is the first mean value theorem or Lagrange's Theorem

2\. Choosing $a=0$ in Taylor's Theorem, we get

$$f(h) = f(0) + \frac{h}{1!} f(0) + \cdots + \frac{h^n}{n!} f^{(n)}(\theta h)$$

$$0 < \theta < 1$$

which is called Maclaurin's Theorem with Lagrange's form of remainder

3\. Writing $a+h = x$. or $h = x-a$ in Taylor's Theorem, we get

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^n(a+\theta(x-a))$$

It can be proved that $R_n \longrightarrow 0$ as $n \longrightarrow \infty$

$\therefore$ $f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \infty$

which is called the Taylor's series of $f(x)$ about the point $x=a$.

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a)}$$

In particular, if $a=0$, we get

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)}$$

which is called the Maclaurin's series of $f(x)$

Exercises

1. Prove that $\frac{1}{1-x} = \frac{1}{3} + \frac{(x+2)}{3^2} + \frac{(x+2)^2}{3^3} + \frac{(x+2)^3}{3^4} + \dots \infty$

$f(x) = \frac{1}{1-x}$; $a = -2$

$\cancel{f'(x) = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}}$

Term 1: $f(a) = f(-2) = \frac{1}{1+2} = \frac{1}{3}$

Term 2: $f'(a) \cdot \frac{(x-a)}{1!}$

$f'(x) = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$

$f'(-2) = f'(a) = \frac{1}{(1+2)^2} = \frac{1}{3^2}$

term 2 $= \frac{1}{3^2}(x+2)$

Term 3: $f''(a) \frac{(x-a)^2}{2!}$

$f''(a) = (-2)\left(\frac{1}{1-x}\right)^3(-1) = \frac{2}{(1-x)^3}$

$f''(-2) = f''(a) = \frac{2}{3^3}$

term 3 $= \frac{\cancel{2}}{3^3} \cdot \frac{(x+2)^2}{\cancel{2!}} = \frac{(x+2)^2}{3^3}$

Term 4: $f'''(a) = \frac{(2)(-3)(-1)}{\cancel{x}(1-x)^4} = \frac{6}{(1-x)^4}$.

$f'''(-2) = \frac{6}{3^4}$

term 4: $f'''(a) \cdot \frac{(x-a)^3}{3!}$

$= \frac{\cancel{6}}{3^4} \frac{(x+2)^3}{\cancel{3!}} = \frac{(x+2)^3}{3^4}$.

$\therefore \frac{1}{1-x} = \frac{1}{3} + \frac{(x+2)}{3^2} + \frac{(x+2)^2}{3^3} + \frac{(x+2)^3}{3^4} + \dots \infty$

2. Expand $\ln x$ in powers $(x-1)$ and hence evaluate $\ln(1.1)$, correct to four decimal places.
(4 terms : 3rd derivatives)

According to Taylor's ~~Theorem~~ series:

$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \infty$

$a = 1$; $x - a = x - 1$

$f(a) = f(1) = \ln 1.$
$= 0$

$f(x) = \ln x.$

$f'(x) = \frac{1}{x}$ $\qquad f'(a) = f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$ $\quad f'''(x) = 2/x^3$ $\qquad f''(a) = f''(1) = -1$

$f'''(a) = f'''(1) = 2$

$$f(x) = 0 + \frac{(x-1)}{1!} - \frac{(x-1)^2}{2!} + \frac{(x-1)^3(2)}{3!}$$

To evaluate $\ln(1.1)$, substituting $x = 1.1$

$$\ln(1.1) = \frac{(1.1-1)}{1!} - \frac{(1.1-1)^2}{2!} + \frac{(1.1-1)^3}{3!}$$

$$= 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3}$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3}$$

$$= \frac{0.6}{6} - \frac{0.03}{6} + \frac{0.002}{6}$$

$$= \frac{0.602 - 0.030}{6} = \frac{0.572}{6}$$

$$= 0.0953$$

```
     0.09533
 6 | 0.57200
     54
     32
     30
     20
     18
     20
```

$\ln(1.1) = 0.0953$

3. Prove that $\ln(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots \infty$

(Maclaurin series ; $a=0$).

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \infty$$

$f(x) = \ln(1+\sin x)$ $\quad f(0) = \ln(1) = 0$.

$f'(x) = \frac{\cos x}{1+\sin x}$ $\quad f'(0) = 1$

$$f''(x) = \frac{(-\sin x)(1+\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{\cancel{8} - \sin^2 x - \cos^2 x - \sin x}{(1+\sin x)^2} = \frac{-1-\sin x}{(1+\sin x)^2}$$

$f''(x) = \frac{-1}{1+\sin x}$ $\quad f''(0) = -1$

$$f'''(x) = \frac{(-1)(-1)}{(1+\sin x)^2} \cdot \cos x.$$

$f'''(x) = \frac{\cos x}{(1+\sin x)^2}$ $\quad f'''(0) = 1$

$$\begin{cases} f(x) = 0 + x + \frac{x^2}{2}(-1) + \frac{x^3}{3!}(1) + \cdots \\ = x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \infty \end{cases}$$

$f''''(x) = (-\sin x)(1+\sin x)^2 - (\cos^2 x)(2)(1+\sin x)$

Method 2:

$$y = f(x) = \ln(1+\sin x).$$

$$1+\sin x = e^{y} \quad \text{—— (1)}$$

At $x=0$,

$$1 = e^{y(0)} \Rightarrow \boxed{y(0)=0}.$$

Differentiating (1) wrt x.

$$\cos x = e^{y}\frac{dy}{dx} = e^{y} y_1 \quad \text{—— (2)}$$

At $x=0$

$$1 = (1)\, y_1(0) \Rightarrow \boxed{y_1(0)=1}$$

Differentiating (2) wrt x.

$$-\sin x = e^{y} y_1^2 + e^{y} y_2 \quad \text{—— (3)}$$
$$= e^{y}(y_1^2 + y_2)$$

At $x=0$

$$0 = 1 + y_2 \Rightarrow \boxed{y_2=-1}$$

Differentiating (3) wrt x.

$$-\cos x = e^{y}(y_1+y_2) + e^{y}(y_2+y_3)$$
$$= -\sin x + e^{y}(y_2+y_3)$$

At $x=0$

$$-1 = 0 + 1(-1 + y_3(0))$$

$$-1 = -1 + y_3(0) \Rightarrow \boxed{y_3=0}$$

$\cancel{-\cos x = e^{y}(y_1 + y_2) + e^{y}(y_2 y_1 + y_3 y_1)}$

$\cancel{-\cos x = e^{y} y_1 + e^{y} y_2 y_1 + e^{y} y_2 + e^{y} y_3 y_1}$

$-\cos x = e^{y} 2y_1 y_2 + e^{y} y_1^3 + e^{y} y_2 y_1 + e^{y} y_3$

$-\cos x = e^{y} y_3 + 3e^{y} y_1 y_2 + e^{y} y_1^3 \quad \text{—} \quad (4)$

At $x=0$

$$-1 = y_3(0) + 3(-1) + 1$$

$$\boxed{1 = y_3(0)}$$

Diff (4) wrt x.

$$\sin x = e^{y} y_3 y_1 + e^{y} y_4 + 3e^{y} y_1 y_2 y_1 + 3e^{y}(y_2 y_2 + y_1 y_3) + e^{y} y_1 y_1^3 + e^{y} 3y_1^2 y_2$$

At $x = 0$

$$0 = 0 + y_4(0) + 3(-1) + 3(1+1) + 1 + \cancel{3(-1)} - 3$$

$$0 = y_4(0) - 3 + 6 + 1 - 3 = +2$$

$$y_4(0) = -2$$

4. Evaluate $\sqrt{25.15}$ using Taylor's Theorem

let $f(x) = \sqrt{x}$.

let $a = 25$.

$f(x) = \sqrt{x}$ $\quad\quad f(a) = 5$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad\quad f'(a) = \frac{1}{10}$

$f''(x) = \frac{1}{2} \cdot \frac{-1}{2} x^{-3/2}$

$= \frac{-1}{4 \cdot x\sqrt{x}}$ $\quad\quad f''(a) = \frac{-1}{4 \times 25 \times 5}$

$\boxed{f''(a) = \frac{-1}{500}}$

Taylor series

$$f(x) = f(a) + \frac{(x-a) f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \ldots$$

$x = 25.15$, $a = 25$.

$$f(25.15) = 5 + \frac{0.15}{10} + \frac{(0.15)^2}{2}\left(\frac{-1}{500}\right)$$

$$= 5 + 0.015 + \frac{-0.0225}{1000}$$

$$= 5 + 0.015 - 0.0000225$$

$$= 5.015 - 0.0000225$$

$\sqrt{25.15} = 5.0149775 \Rightarrow$ $\boxed{\cancel{\sqrt{25.15} = 5.015}}$

5. Find the value of $\tan 43^\circ$ using Taylor series :

Let $f(x) = \tan x$ $\quad a = \pi/4.$

$f(a) = 1$

$f'(x) = \sec^2 x$ $\quad f'(a) = 2$

$f''(x) = 2\sec^2 x \tan x$ $\quad f''(a) = 2 \times 2 \times 1 = 4$

Taylor series

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$\tan x = 1 + (x - \pi/4)\, 2 + \frac{(x-\pi/4)^2}{2} \cdot 4$$

$$= 1 + (x-\pi/4) \times 2 + (x-\pi/4)^2 \times 2$$

$$(x-a) = \frac{(43-45)\pi}{180} = \frac{-\pi}{90}$$

$$\tan 43^\circ = 1 + \frac{(-2) \times \pi}{90} + \frac{\pi^2}{90^2} \times 2$$

$$= 1 - \frac{\pi}{45} + \frac{2\pi^2}{8100}$$

$$= 1 - 0.06981 + 0.0024369$$

$$= \cancel{1+0} \; 1.0024369 - 0.06981$$

$$\boxed{\tan 43^\circ = 0.9326}$$

6. Find $\cosh 1.505$, given $\sinh(1.5) = 2.1293$, $\cosh(1.5) = 2.3524$

Let $f(x) = \cosh x$ $\qquad a = 1.5.$

$f(x) = \cosh x$ $\qquad f(a) = 2.3524$

$f'(x) = \sinh x$ $\qquad f'(a) = 2.1293$

$f''(x) = \cosh x$ $\qquad f''(a) = 2.3524.$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(x) + \frac{(x-a)^2}{2!} f''(x) + \dots$$

1^{st} derivative sufficient.

$$= 2.3524 + \frac{(0.005)(2.1293)}{1} + \frac{(0.000025)(2.3524)}{2}$$

$= \cancel{2.3524} + \cancel{0.00532}$

$= \cancel{2.3}$

$= 2.3524 + 0.0106465 + 0.0000294$

$= 2.363076.$

7. Expand $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ as a Maclaurin series. (5th degree terms).

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

$$\tan y = \frac{\sqrt{1+x^2}-1}{x}$$

let $x = \tan\theta.$

$dx = \sec^2\theta \, d\theta$

$$\tan y = \frac{\sec\theta - 1}{\tan\theta} = \operatorname{cosec}\theta - \cot\theta.$$

At $x=0$, $\tan\theta = 0 \Rightarrow \theta = 0$.

$\tan y(0) = \operatorname{cosec}\theta - \cot\theta \rightarrow$ (1).

$\tan y(0) = \frac{\sqrt{1+x^2}-1}{x}$ at $x=0$.

$\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x} = \lim_{\theta\to 0} \frac{\sec\theta - 1}{\tan\theta}$

$= \lim_{\theta\to 0} \frac{\cancel{\sec\theta}\tan\theta}{\sec^2\theta} = \cancel{\operatorname{cosec}\theta}.\sin\theta$

$= 0$.

$\tan y = 0 \Rightarrow \boxed{y(0) = 0}$

Diff. (1) wrt θ

$\cancel{\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}}$.

$\cancel{\sec^2 y \cdot \frac{dy}{dx} = -\operatorname{cosec}\theta\cot\theta \frac{d\theta}{dx} + \operatorname{cosec}^2\theta \frac{d\theta}{dx}}$.

$\cancel{(1)\frac{dy}{dx} =}$

$\tan y = \frac{\sqrt{1+x^2}-1}{x}$.

$\sec^2 y\; y_1 = \frac{x\left(\frac{2x}{2\sqrt{1+x^2}}\right) - (\sqrt{1+x^2}-1)}{x^2}$

$= \frac{\frac{x^2}{\sqrt{1+x^2}} - (\sqrt{1+x^2}-1)}{x^2}$.

$$\tan y = \frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}} = \frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2} = \tan\theta/2$$

$$y = \frac{\theta}{2} \Rightarrow 2y = \tan^{-1}x$$

$$\boxed{y = \frac{1}{2}\tan^{-1}x} \rightarrow 1$$

$y = f(x)$

$$f(x) = \frac{1}{2}\tan^{-1}x \qquad f(0) = 0$$

$$f'(x) = \frac{1}{2(1+x^2)} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{1}{\cancel{2}}\frac{(-1)\cdot\cancel{2}x}{(1+x^2)^2} \qquad f''(0) = \frac{0}{(1+0)^2}$$

$$= \frac{-x}{(1+x^2)^2} \qquad = 0$$

$$f'''(x) = \frac{(-1)(1+x^2)^2 + (x)(2(1+x^2))(2x)}{(1+x^2)^4}$$

$$= \frac{-1(1+x^2)^2 + 4x^2(1+x^2)}{(1+x^2)^4}$$

$$= \frac{-1(1+x^4+2x^2) + 4x^2+4x^4}{(1+x^2)^4}$$

$$= \frac{-1-x^4-2x^2+4x^2+4x^4}{(1+x^2)^4}$$

$$= \frac{3x^4+2x^2-1}{(1+x^2)^4}$$

when $x=0$

$\tan 2y = 0$

$y=0$

$\boxed{y(0)=0}$

$\tan 2y = x$

Differentiating wrt x.

$$\sec^2 2y \cdot 2\frac{dy}{dx} = 1$$

$$2\frac{dy}{dx}\cdot \sec^2 2y = 1 \longrightarrow (1).$$

$$\frac{dy}{dx} = \frac{1}{2}\cos^2 2y = \frac{1}{2}\times 1$$

$$\boxed{y_1(0) = 1/2}$$

Diff. (1) wrt x.

$$(2) \longleftarrow 2(y_2 \cdot \sec^2 2y + y_1 \cdot 2\sec^2 2y \tan 2y \, y_1) = 0.$$

$$y_2(1) + \frac{1}{2}\times 2\times 1 \times 0 = 0.$$

$$\boxed{y_2 = 0}$$

$$y_2 \sec^2 2y + (y_1)^2 \sec^2 2y \tan 2y = 0 \quad \rightarrow (2)$$

Diff (2) wrt x.

$$y_3 \sec^2 2y + y_2 \cdot 2\sec^2 2y \tan 2y \, y_1 2$$
$$+ 2y_1 y_2 (\sec^2 2y \tan 2y)$$
$$+ (y_1)^3 (2\sec^2 2y \tan 2y + \sec^4 2y) \times 2$$
$$= 0$$

$$y_3 (1) + 0 + 0 + \left(\frac{1}{2}\right)^3 \times (2 \times 0 + 1) \times 2 = 0$$

$$y_3 + \frac{1}{8} \times 4 = 0$$

$$\boxed{y_3 = -1/2}$$

Shorter method.

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right) = \frac{1}{2} \tan^{-1} x.$$

$$\boxed{y(0) = 0}.$$

$$y_1 = \frac{1}{2} \cdot \frac{1}{1+x^2}$$

$$\boxed{y_1(0) = \frac{1}{2}.}$$

$$(1+x^2)y_1 = \frac{1}{2} \longrightarrow ①$$

- Leibnitz's Theorem for differentiation of a product n times

$$\frac{d^n}{dx^n}(uv) = uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + \dots + {}^nC_{n-1} u_{n-1} v_1 + {}^nC_n u_n v$$

$$\frac{d^n}{dx^n}(uv) = \sum_{r=0}^{n} {}^nC_r u_r v_{n-r}$$

Applying Leibnitz's Theorem for diff. on ①.

~~for the 1st derivative.~~

$u = (1+x^2)$ $\qquad v = y_1$

To find the n^{th} derivative.

$$\frac{d^n}{dx^n}\big(\underset{u}{(1+x^2)}\underset{v}{y_1}\big) = \cancel{(1+x^2)y} \frac{d^n}{dx^n}(1/2)$$

$$\frac{d^n}{dx^n}(uv) = (1+x^2)y_{n+1} + {}^nC_1(2x)y_n + {}^nC_2(2)y_{n-1} + {}^nC_3\overset{0}{\cancel{(0)}}(y_{n-2}) = 0.$$

$$\boxed{\frac{d^n}{dx^n}(uv) = (1+x^2)y_{n+1} + n(2x)y_n + {}^nC_2(2)(y_{n-1}) = 0}$$

~~(1) For $n=0$,~~

~~$y(0) = 0$~~

~~(2) For $n=1$~~

~~$(1+x^2)y_{1+1} + (1)(2x)(y_1) + \ldots$~~

~~$(1+x^2)y_{1+1} + {}^1C_1(2x)(y_1) = 0$~~

At $x=0$,

$$(1+x^2)\,y_{n+1} + 2nx\,y_n + n(n-1)\,y_{n-1} = 0$$

$$y_{n+1}(0) + n(n-1)\,y_{n-1}(0)$$

recurrence value: $\boxed{y_{n+1}(0) = -n(n-1)\,y_{n-1}(0)}$

for $n=1,2,3$.

$n=1 \Rightarrow$

$$y_2(0) = -1(1-1)\,y_1(0)$$

$$\boxed{y_2(0) = 0}$$

~~$n=2$~~

~~$y_3(0) = -2(2-1)\,y$~~

$\therefore\ y_2(0) = 0,\ y_4 = 0,\ y_6 = 0 \ldots$

$y_{2n}(0) = 0\ \forall n$

$\because$ if $\cancel{y_{n+1}}$ $y_{n-1} = 0$, $y_{n+1} = 0$.

for $n = 3$, $y_2 = 0 \Rightarrow y_4 = 0$

$\vdots$

$\cancel{y =}$ $y(0) = 0$

$y_1(0) = 1/2$

$y_2(0) = 0$

$n = 2$

$y_3(0) = -2(2-1)\, y_1(0)$

$= -2 \times 1/2 = \cancel{1/2}\ -1$

$\boxed{y_3(0) = -1}$

$n = 3 \Rightarrow \boxed{y_4(0) = 0}$

$n = 4$.

$y_5(0) = -4(4-1)\, y_3(0)$

$= -4 \times 3 \times \cancel{1/2}\ -1 = \cancel{6}\ 12$

$\boxed{y_5(0) = 12}$

$n = 6$

$y_7(0) = -6(5)\, y_5$

$y_7(0) = -6 \times 5 \times 12$

$y_7(0) = \cancel{180}\ -360$.

By Maclaurin's Series.

$$f(x) = f(0) + \frac{x}{1!} f'(x) + \frac{x^2}{2!} f''(x) + \frac{x^3}{3!} f'''(x) + \dots$$

$$y(x) = y(0) + \frac{x}{1!} y_1 + \frac{x^2}{2!} y_2 + \frac{x^3}{3!} y_3 + \dots$$

$$\cancel{\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{x}{2} - \frac{x^3}{3!} + \frac{x^5}{5!}}$$

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \infty\right)$$

8. Expand $e^{a\sin^{-1}x}$ as a Maclaurin series as far as 5^{th} degree terms.

$$y = e^{a\sin^{-1}x}$$

$$\ln y = a\sin^{-1}x$$

$$\boxed{y(0) = e^0 = 1}$$

$$y_1 = \frac{a e^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\boxed{y_1(0) = a}$$

$$\frac{d^n}{dx^n}(uv) = uv_n + {}^nC_1\, u_1 v_{n-1} + {}^nC_2\, u_2 v_{n-2} + \dots + {}^nC_n\, u_n v$$

$$y_1 = \frac{a\, e^{a\sin^{-1}x}}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

$$(y_1)^2 = \frac{a^2y^2}{1-x^2}$$

$$(y_1)^2(1-x^2) = a^2y^2$$

Diff. wrt x.

$$2\cancel{(y_1)}(y_2)(1-x^2) + (y_1)^{\cancel{2}}(-2x) = a^2\cdot 2y\cancel{y_1}$$

$$\cancel{2}y_2(1-x^2) - \cancel{2}xy_1 = a^2\cancel{2}y$$

$$\boxed{y_2(1-x^2) - xy_1 = a^2y} \longrightarrow ①$$

At $x=0$,

$$y_2 - 0 = a^2$$

$$\boxed{y_2 = a^2}$$

$$\overset{A}{\underbrace{y_2(1-x^2)}} = \overset{B}{\underbrace{xy_1}} + \overset{C}{\underbrace{a^2y}} \longrightarrow ②$$

Diff. wrt x on both sides, n times.

For A:

$y_2(1-x^2)$: $u = 1-x^2$ $v = y_2$

$$\frac{d^n}{dx^n}\left(y_2(1-x^2)\right) = (1-x^2)\, y_{n+2} + {}^nC_1(-2x)\, y_{n+1} + {}^nC_2(-2)\, y_n$$

$$= (1-x^2)\, y_{n+2} + {}^nC_1(-2x)\, y_{n+1} + {}^nC_2(-2)\, y_n$$

For B:

xy_1 $u = x$ $v = y_1$

$$\frac{d^n}{dx^n}(xy_1) = x y_{n+1} + {}^nC_1 y_n$$

For C:

a^2y

$$\frac{d^n}{dx^n}(a^2y) = a^2 y_n$$

~~Diff. of (2)~~ is

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$$(1-x^2)(y_{n+2}) - 2nx(y_{n+1}) + \frac{n(n-1)}{2}(-2)y_n$$
$$= +xy_{n+1} + ny_n + a^2 y_n$$

$$(1-x^2)(y_{n+2}) - 2nx(y_{n+1}) - n^2 y_n + \cancel{n y_n}$$
$$= xy_{n+1} + \cancel{n y_n} + a^2 y_n$$

$$(1-x^2)(y_{n+2}) - (2n+1)x y_{n+1} - (n^2+a^2)y_n = 0$$

At $x=0$.

$$\boxed{y_{n+2}(0) = (a^2+n^2)y_n(0)} \qquad n = 0, 1, 2 \ldots$$

$$y(0) = 1$$
$$y_1(0) = a$$
$$y_2(0) = a^2$$

For $n=1$

$$y_3(0) = (a^2+1)y_1 = (a^2+1)a$$

$$\boxed{y_3(0) = a^3 + a} = (a^2+1)a$$

$n=2$

$$\boxed{y_4(0) = (a^2+4)a^2}$$

$n=3$

$$y_5(0) = (a^2+9)(a^3+a) = (a^2+9)(a^2+1)(a)$$
$$= a^5 + 9a^3 + a^3 + 9a$$

$$y_5(0) = a^5 + 10a^3 + 9a$$

The Maclaurin series

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$e^{a\sin^{-1}x} = 1 + xa + \frac{x^2}{2} \cdot a^2 + \frac{x^3}{3!}(a)(a^2+1)$$

$$+ \frac{x^4}{4!}(a^2)\cancel{(a^2+1)}(a^2+4)$$

$$+ \frac{x^5}{5!}(a)(a^2+1)\cancel{(a^2+4)}(a^2+9) + \dots + \infty$$

9. In the MVT, $f(x+h) = f(x) + hf'(x+\theta h)$. Show that $\theta = 1/2$ for $f(x) = ax^2+bx+c$ in $(0,1)$

$$f'(x) = 2ax + b$$

$$\cancel{a(x+h)^2 + b(x+h) + c =}$$

$f(x) = ax^2 + bx + c$
$f'(x) = 2ax + b$

$$a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + h(2a(x+\theta h) + b)$$
$$\cancel{ax^2} + ah^2 + \cancel{2axh} + \cancel{bx} + \cancel{bh} + \cancel{c} = \cancel{ax^2} + \cancel{bx} + \cancel{c} + h(\cancel{2ax} + 2a\theta h + \cancel{b})$$

$$ah^2 + 2axh = 2ahx + 2a\theta h^2$$

$$\cancel{ah^2 + 2ax - 2ahx = 2a\theta h^2}$$

$$\cancel{h^2 + 2x - 2hx = 2\theta h^2}$$

$$\cancel{a}\cancel{h^2} = 2\cancel{a}\theta \cancel{h^2}$$

$$\boxed{\theta = 1/2}$$

29-08-19

10- Find the first 3 terms and the Legrange's remainder of the function $e^{ax} \sin bx$.

(Maclaurin's)

$$f(x) = e^{ax} \sin bx = y$$

Taylor expansion

$$\cancel{f(a+h) = f(a) + (x-a)}$$

$$\cancel{f(x) = f}$$

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^{(n)}(a + \theta h)$$

$$f'(x) = ae^{ax} \sin bx + be^{ax} \cos bx = y_1$$

$$\boxed{f'(0) = b}$$

let $a = r\cos\alpha$, $b = r\sin\alpha$, $r = \sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\frac{b}{a}$

$$f'(x) = re^{ax} \cos\alpha \sin bx + re^{ax} \sin\alpha \cos bx$$

$$f'(x) = re^{ax} (\sin(bx + \alpha))$$

$$\cancel{f''(x)} = f'(0) = r\sin\alpha = b.$$

$$f''(x) = rae^{ax}\sin(bx+\alpha) + rbe^{ax}\cos(bx+\alpha)$$

$$= re^{ax}(a\sin(bx+\alpha) + b\cos(bx+\alpha))$$

$$= re^{ax}(r\cos\alpha \sin(bx+\alpha) + r\sin\alpha\cos(bx+\alpha))$$

$$= r^2e^{ax}(\sin(bx+2\alpha))$$

$$\boxed{f''(x) = r^2e^{ax}\sin(bx+2\alpha)}$$

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

$$\tan\alpha = b/a$$

$$f''(0) = r^2\sin 2\alpha$$

$$= (a^2+b^2)\left(\frac{2b/a}{1+b^2/a^2}\right)$$

$$= \frac{\cancel{(a^2+b^2)}(2b)(a^{\cancel{2}})}{\cancel{(a)}\cancel{(a^2+b^2)}}$$

$$\boxed{f''(0) = 2ab}$$

$$f^{(n)}(x) = r^n e^{ax}\sin(bx+n\alpha)$$

~~Lagrange's remainder:~~

$$f'''(x) = r^3e^{ax}\sin(bx+3\alpha)$$

$$f'''(0) = r^3\sin(3\alpha)$$

$$= r^3(3\sin\alpha - 4\sin^3\alpha)$$

$$= r^3(3\sin\alpha - 4\sin^3\alpha)$$

$$= (a^2+b^2)^{3/2} \cdot \left(3 \cdot \frac{b}{r} - 4\frac{b^3}{r^3}\right)$$

$$= \frac{\cancel{(a^2+b^2)^{3/2}}}{\cancel{(a^2+b^2)^{3/2}}}\left(3br^2 - 4b^3\right)$$

$$= 3br^2 - 4b^3 = 3b(a^2+b^2) - 4b^3$$

$$\boxed{f'''(0) = 3ba^2 - b^3}$$

$f(0) = 0$

$f'(0) = b$

$f''(0) = 2ab$

$f'''(0) = 3ba^2 - b^3$

3 terms

<u>Lagrange's remainder:</u>

$$f^{(n)}(x) = r^n e^{ax} \sin(bx + n\alpha)$$

$$f^{(n)}(x) = (a^2+b^2)^{n/2} e^{ax} \sin(bx + n\tan^{-1} b/a)$$

At $x = 0$.

$$\cancel{f^{(n)}(0) = r^n \sin(n\alpha) = (a^2+b^2)^{n/2} \sin\left(n \tan^{-1}\frac{b}{a}\right)}$$

$$f^{(n)}(0) = (a^2+b^2)^{n/2} \sin\left(n \tan^{-1}\frac{b}{a}\right)$$

By Maclaurin's Theorem,

$$\cancel{f(x) = f(a) + \frac{(x-a)}{1!} f'(0)}$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$$

$$+ \dots + \frac{x^n}{n!} f^{(n)}(\theta x)$$

$$f(x) = xb + x^2 ab + \frac{x^3(3ba^2 - b^3)}{6} + \dots +$$

$$\frac{x^n}{n!}\left[(a^2+b^2)^{n/2} e^{a\theta x} \sin(b\theta x + n\tan^{-1} b/a)\right]$$

11. Find the Maclaurin's series of the following functions.

(a) e^x (b) $\ln(1+x)$ (c) $\ln(1-x)$ (d) $\ln\left(\frac{1+x}{1-x}\right)$

(e) $\sin x$. (f) $\cos x$ (g) $\tan x$ (h) $\sinh x$

(i) $\cosh x$ (j) $\tan^{-1} x$

5th degree term

Maclaurin's series

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \frac{x^5}{5!} f'''''(0) + \dots$$

(a) $f(x) = e^x$

$f'(x) = e^x$ $f^{(n)}(x) = e^x$

$f(0) = 1$ $f^{(n)}(x) = 1$

$$\boxed{e^x = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots}$$

(b) $\ln(1+x) = f(x)$ $f(0) = 0$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ $f'(0) = 1$

$f''(x) = -1(1+x)^{-2}$ $f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$ $f'''(0) = 2$

$f^4(x) = -6(1+x)^{-4}$ $f^4(0) = -6$

$$f^5(x) = 24(1+x)^{-5} \qquad f^5(0) = 24$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots}$$

(c) $\ln(1-x)$; substitute $x = -x$ in (b)

$$\boxed{\ln(1\cancel{+}x) = -x - \frac{x^2}{2!} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \dots}$$

(d) $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$= (b) - (c)$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \dots$$

$$\boxed{\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \dots\right)}$$

(e)

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^4(x) = \sin x \qquad f^4(0) = 0$

$\vdots \qquad \vdots$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots}$$

(f) $f(x) = \cos x$ $\quad f(0) = 1$

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$f^2(x) = -\cos x$ $\quad f^2(0) = -1$

$f^3(x) = \sin x$ $\quad f^3(0) = 0$

$f^4(x) = \cos x$ $\quad f^4(0) = 1$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots}$$

(g) $f(x) = \tan x = y$ $\quad y(0) = 0$

$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2$

$y_1 = 1 + y^2$ $\quad \boxed{y_1(0) = 1}$

$y_2 = 2y_1 y$ $\quad \boxed{y_2(0) = 0}$

$y_3 = 2y_2 y + 2(y_1)^2$ $\quad \boxed{y_3(0) = 2}$

$y_4 = 2(y_3 y + (y_2)^2)$ $\quad y_4(0) = 2(0) + 4(0)$

$+ 2 \cdot 2 y_1 y_2$ $\quad \boxed{y_4 = 0}$

$\cancel{y_5 = 2(y_4 y + 2y_2 y_3)}$

$y_5 = 2(y_4 y + y_3 y_1 + 2y_2 y_1)$ $\quad y_5(0) = 2(2) + 4(2)$

$+ 4(y_2^2 + y_1 y_3)$ $\quad \boxed{y_5(0) = 12}$

$$\tan x = x + \frac{2x}{3!} + \frac{12x^5}{5!} + \dots$$

$$= \cancel{x +} \frac{\cancel{x}}{\cancel{3}} \quad x + \frac{x}{3} + \frac{x^5}{10} + \dots$$

$$\boxed{\tan x = x + \frac{x}{3} + \frac{x^5}{10} + \dots}$$

(h) $f(x) = \sinh x$ $\quad f(0) = (0)$

$f'(x) = \cosh x$ $\quad f'(0) = 1$

$f''(x) = \sinh x$ $\quad f''(0) = 0$

$\vdots$

$$\boxed{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots}$$

(i) $f(x) = \cosh x$ $\quad f(0) = 1$

$f'(x) = \sinh x$ $\quad f'(0) = 0$

$f''(x) = \cosh x$ $\quad f''(0) = 1$

$\vdots$

$$\boxed{\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots}$$

(j) $f(x) = \tan^{-1} x = y$

$\tan y = x.$ $\quad \boxed{y(0) = 0}$

$\cancel{\sec^2 y \; y_1 = x}$ $\quad \cancel{y_1(0) = 0}$

$\sec^2 y \; y_1 = 1$ $\quad \boxed{y_1(0) = 1}$

$(1 + \tan^2 y)\, y_1 = 1 = y_1(1 + x^2)$

$$1 = y_1(1+x^2) \longrightarrow ①$$

Leibnitz Theorem:

$$\frac{d^n}{dx^n}(uv) = \sum_{r=0}^{n} {}^nC_r\, u_r\, v_{n-r}$$

$u = 1+x^2 \qquad v = y_1$

$$\frac{d^n}{dx^n}(uv) = uv_n + n\,u_1 v_{n-1} + {}^nC_2\, u_2 v_{n-2}$$

$$\frac{d^n}{dx^n}(uv) = (1+x^2)\,y_{n+1} + n(2x)\,y_n + {}^nC_2(2)(y_{n-1})$$

For $x = 0$

$$\frac{d^n}{dx^n}(uv) = y_{n+1} + n(n-1)\,y_{n-1}$$

~~n=3~~

$$\cancel{\frac{d^3}{dx^3}\left(y_1(1+x^2)\right) = y_4 + 3(2)y_2}$$

$n=1$

$$\frac{d}{dx}\left(y_1(1+x^2)\right) = y_2 + 0 = 0$$

$y_2 = 0 \qquad \boxed{y_2(0) = 0}$

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$ — store 67 arc length

$n = 2$

$$\frac{d^2}{dx^2}(y_1(1+x^2)) = y_3 + 2y_1 = 0$$

$$y_3 + 2 = 0$$

$$\boxed{y_3(0) = -2}$$

$n = 3$

$$y_4 + 6y_2 = 0$$

$$\boxed{y_4(0) = 0}$$

$n = 4$

$$y_5 + 12y_3 = 0$$

$$\boxed{y_5 = 24}$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$= 0 + x + \frac{x^2}{2!} \times 0 + \frac{x^3}{6}(-2) + 0 + \frac{x^5}{5}$$

$$\boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots}$$